Imperial College London
of Science, Technology and Medicine
Department of Physics

# The Ill Early Universe

Selim Can Hotinli

*…With thanks to Professor Andrew Jaffe for his support and guidance for the work done in this paper.*

*to dad & to mom ,*

*for their endless support…*

## Abstract

In this work, we consider condensed matter phenomena in the early Universe. We focus mainly on the effects of quenched disorder during inflation. We draw analogies between early Universe cosmology and emergent phenomena. We suggest considering disorder as a classically sourced effect on wave-modes around the horizon crossing. Improving previous work in this area, we apply a tool from condensed matter physics to early Universe cosmology and calculate corrections to the two-point correlation function for fluctuations. We then turn our attention to the microscopic physics of the early Universe and suggest non-equilibrium quantum field theory formalism for the study of disorder in the early Universe.

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# Chapter 1

# Introduction

The success of observational cosmology in the past decade has promoted this area to what may soon be called precision science. With the recent results from Planck satellite; most cosmological parameters are now measured up to order of few percents of error [1]. Moreover, *Planck* showed that the early Universe is very isotropic and homogeneous with only very little deviations, on the order of $\mathcal{O}(10^{-5})$. However, there is still much we do not understand about the physics of the early Universe that gave rise to this isotropy and homogeneity we now measure. Much effort has been made in the past few decades to improve our understanding of this era. With the ever increasing scientific precision of observational cosmology, it is undeniably a very exciting time to study the early Universe as it might not be too distant in the future that we begin getting some answers. In what follows, we will be motivated with these ideas.

Perhaps one of the few things that is well understood about early Universe is that it necessarily involves extreme events that ultimately lead to the Universe we now live in. In this work, we will focus on the most extreme of these, *inflation*. Although inflation has been essential in our better understanding of early Universe, there is still much about it that is speculative. It is suggestive, however, that this is not due to our lack of theoretical understanding but more likely due to our limitations in observations. From a phenomenological perspective, the study of inflation allows us to formulate and predict distinct signatures of the primordial dynamics that is manifest in our observations of the early Universe and also the large scale structure. There has been many studies on inflation with ever increasing range of applications and sophistication. Here, we will make an attempt towards adding to our current understanding of inflationary phenomena through considering analogies between condensed matter physics and early Universe cosmology.

Searching for dualities between condensed matter and other branches of physics is in fact quite common in modern theoretical research. This is perhaps even more apparent from the point of view of the complexity science. We define complexity as the study of large scale behaviour of complicated dynamics via universal characteristics and emergent phenomena. In this paper, complexity reflects to our analysis of the early Universe as studying the phenomena such as localisation and percolation along with coarse-grained statistical effects due to underlying fundamental physics. One such mechanism that we focus throughout the paper is *disorder*. Disorder is a very general phenomena and especially the study of *quenched* disorder in the past few decades has largely shaped the contemporary research in condensed matter physics and many other areas of science. However, in early Universe cosmology, it is possible that this phenomena is somewhat overlooked. This is perhaps even more apparent considering inflation. In this paper, we take a step towards studying disorder in the early Universe from the perspective of condensed matter physics.

Due to the suggestive inadequacy of the available tools in early Universe cosmology, our quantitative efforts in this paper consists mainly of introducing and reviewing various formalisms with which one can study disorder. Condensed matter physics provides a collection of strong tools well developed for analysing a wide variety of systems and their dynamics. Depending on the details of the disorder phenomena, the dynamics of a condensed matter system may show distinctive properties. Here, we give attention to the *quenched* characteristic of disorder. Quenching leads us to consider methods beyond the perturbation theory as it can drive a system out-of equilibrium. In recent years, the studies of non-equilibrium systems at extreme conditions has gained much attention in the condensed matter community. In early Universe cosmology, a direct analogue of these systems can be found in the study of the phenomena following inflation [2, 3]. In this paper we also will consider the effects of disorder *during* inflation.

In Chapter §2 we begin with a pedagogical review of the relevant methods and techniques in modern cosmology. There, we introduce the flat FRW universe, review the dynamics of inflation and then introduce cosmological perturbation theory. In Section §2.5 we introduce the effective field theory (EFT) of inflation. The EFT of inflation plays a central role in establishing analogies between the early Universe cosmology and condensed matter physics. We discuss this reasoning in Chapter §3 where we also review some of the methods available to condensed matter physics. There, we focus on disorder in relation to the emergent phenomena and discuss the contemporary efforts in condensed matter physics. We conclude Chapter §3 by introducing the so called replica field theory developed for calculating statistical properties of disordered classical systems. In Chapter §4 we begin by deriving a general expression for the correlation functions of a system with disorder. Next in Section §4.2, we consider classical applications of the replica field theory method to de Sitter spacetime and to the super-horizon dynamics during inflation. There, we calculate the corrections to the two-point correlation functions of the scalar field driving inflation. In addition to our calculations for the scalar field, we also apply this method to EFT of inflation. Lastly in Section §4.3, we take a step towards studying the effects of disorder on sub-horizon microscopic physics during inflation. There, we first review the perturbative treatment of disorder using the EFT formalism. We then take a look into the cosmological *in-in* path integral description in order to go beyond the perturbation theory. We conclude the review of quantum field theory extension of the applications of disorder in the early Universe by reviewing the non-equilibrium quantum field theory; effective action formalism. We give a conclusion in Chapter §5.

# Chapter 2

# Cosmology

In this chapter we will review some of the topics in modern cosmology. We will be following most closely [4–6].

## 2.1 The Homogeneous Universe

### 2.1.1 FRW Spacetime

Following translational invariance (*homogeneity*) and rotational invariance (*isotropy*), one arrives at the Friedmann-Robertson-Walker (FRW) metric for the spacetime of the Universe:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right). \tag{2.1}$$

The expression inside the large brackets, also written as $\mathrm{d}\Sigma^2$, is the spatial metric. The scale factor $a(t)$ characterises the relative size of the three dimensional space of uniform curvature (hypersurface) $\Sigma$ at different times. The curvature parameter $k$ is +1 for positively curved (elliptical) space, 0 for flat (Euclidean) space and −1 for negatively curved (hyperbolic) space. The parameter $\mathrm{d}\Sigma$ is independent of time and all time dependence is on $a(t)$ which expands with the Universe. An important parameter in FRW spacetime is the expansion rate

$$H := \frac{\dot{a}}{a} \tag{2.2}$$

where $H$ is called the *Hubble parameter* and has units of inverse time. This parameter sets the fundamental scale of the FRW spacetime, i.e. the characteristic time and length $t \sim d \sim H^{-1}$ (with natural units $c = 1$). Hubble parameter $H$ is positive (negative) for an expanding (collapsing) universe. Finally, the number of $e$-folds of the expansion is given as

$$N = \int H \mathrm{d}t\ . \tag{2.3}$$

### 2.1.2 Conformal time and horizon

Trajectories of massless photons follow null geodesics, $\mathrm{d}s^2 = 0$, which may be studied most easily by defining a conformal time

$$\tau = \int \frac{\mathrm{d}t}{a(t)}\ . \tag{2.4}$$

With this definition, radial propagation of light in the FRW universe becomes

$$\mathrm{d}s^2 = a(\tau)^2[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2]\ , \tag{2.5}$$

where the metric is now factorised into a static Minkowski metric multiplied by the conformal factor $a(\tau)$. In this representation, the null geodesics of light follows straight lines at ±45° angles in the $\tau-\chi$ plane corresponding to the *light cone*. The largest comoving distance light can travel between an initial time $t_i$ and some later time $t$ is

$$\chi_p(\tau) = \tau - \tau_i \ , \tag{2.6}$$

and it is called the *comoving particle horizon*. The physical size of the particle horizon is

$$d_p(t) = a(t)\chi_p \ . \tag{2.7}$$

Finally the *event horizon* is defined for comoving coordinates that satisfy

$$\chi > \chi_e = \tau_{max} - \tau \ , \tag{2.8}$$

where $\tau_{max}$ is some final time.

### 2.1.3 FRW Dynamics

Dynamics of the Universe is determined by the Einstein Equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \ . \tag{2.9}$$

To calculate the energy momentum tensor of the Universe $T_{\mu\nu}$, we define a timeline velocity 4-vector

$$u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \ , \tag{2.10}$$

where $\tau$ is the proper time, i.e. $g_{\mu\nu}u^\mu u^\nu = -1$. For a perfect fluid equations for $T_{\mu\nu}$ simplifies to give

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu \ , \tag{2.11}$$

where $\rho$ and $p$ are the proper energy density and pressure in fluid rest frame. Choosing a frame that is comoving with the fluid $u^\mu = (1, 0, 0, 0)$, the stress energy tensor becomes

$$T^\mu_\nu = \mathrm{diag}(\rho, -p, -p, -p) \ . \tag{2.12}$$

With this definition, Einstein equations take the form of two coupled non-linear ordinary differential equations called *Friedman Equations*

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2} \qquad \text{and} \qquad \dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \ , \tag{2.13}$$

where over-dots denote derivatives with respect to physical time $t$. For an expanding universe $\dot{a} > 0$ filled with ordinary matter ($\rho + 3p \geq 0$) Eqn. (2.13) implies $\ddot{a} < 0$, indicating a singularity in the finite past $a(t \equiv 0) = 0$. These equations can be combined into a continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + p) = 0 \ , \tag{2.14}$$

which may also be written as

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a} = -3(1 + \omega) \tag{2.15}$$

where we have defined the equation of state parameter

$$\omega := \frac{p}{\rho} \ . \tag{2.16}$$

Integrating Eqn. (2.15) we get

$$\rho \propto a^{-3(1+\omega)} \ . \tag{2.17}$$

Combining this equation with Eqn. (2.13) we get the time evolution of the scale factor

$$a(t) = \begin{cases} t^{2/3(1+\omega)} & \omega \neq -1 \ , \\ e^{Ht} & \omega = -1 \ , \end{cases} \tag{2.18}$$

where we find for the scale factor of a flat universe, i.e. $k = 0$ in Eqn. (2.1) , non-relativistic matter domination ($\omega = 0$) gives $a(t) \propto t^{2/3}$, radiation or relativistic matter domination ($\omega = 1/3$) gives $a(t) \propto t^{1/2}$ and a cosmological constant ($\omega = -1$) gives $a(t) \propto \exp(Ht)$. Cosmological and astronomical observations today tell us that the Universe is flat with $\omega \simeq -1$.

## 2.2 Shortcomings of the Standard Big Bang Theory

The FRW Universe introduced above constitutes to the standard Big Bang theory. While this theory explains many observations very successfully, it fails to give a satisfying understanding of the initial conditions.

### 2.2.1 Horizon Problem

In introducing the FRW formalism we assumed homogeneity and isotropy of the Universe. From large-scale structure (LSS) surveys we know that the Universe today satisfies these conditions on the largest accessible scales. Since inhomogeneities grow in time due to gravity, we would expect the Universe to be more homogeneous as we go back in time. Observations of the cosmic microwave background[1] (CMB) show that this is indeed the case as the inhomogeneities are much smaller at the last-scattering (recombination) surface than they are today. Hence we expect these inhomogeneities to be *remarkably small* at yet earlier times. Moreover we know from the standard Big Bang theory that the early Universe (e.g. observed at the CMB) consists of many causally disconnected regions of space. Consider the *comoving particle horizon*, $\tau$, introduced before as the maximum distance a light ray can travel between an initial time $t_i = 0$ and time $t$. This is equivalent to the fraction of the universe in *causal contact*. Expressing $\tau$ as the integral of the *comoving Hubble radius*, $(aH)^{-1}$

$$\tau \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \frac{\mathrm{d}a}{Ha^2} = \int_0^a \mathrm{d}\ln a \left(\frac{1}{Ha}\right) \ , \tag{2.19}$$

and using the equation of state for the fluid dominated universe given above, we find

$$(aH)^{-1} = H_0^{-1} a^{1/2(1+3\omega)} \ , \tag{2.20}$$

where $H_0$ is the Hubble scale today. During the standard Big Bang expansion ($\omega \geq 0$), $(aH)^{-1}$ grows monotonically and the comoving horizon $\tau$ increase with time. This means that the comoving scales entering the horizon today must have been far outside the horizon at CMB surface. Nevertheless the CMB tells us that the Universe was almost perfectly homogeneous at the time of recombination on scales much larger than what is limited by the causal horizon.

### 2.2.2 Flatness Problem

In addition to the *horizon problem* concerning the initial density distributions, standard Big Bag theory requires fine tuning of initial velocities as well. First, note that the local curvature in the Universe is defined by the difference between kinetic and potential energies. If the initial velocities are too small (or large), universe re-collapses (or expands) rapidly. This results in the failure of forming any structure

[1] See Section §2.6.1 for a review.

in the former case, and becoming nearly empty in the latter cases. Hence the fluid velocities need to be finely tuned across the causally separated regions of space. Consider the previously introduced Friedmann equation,

$$H^2 = \frac{1}{3}\rho(a) - \frac{k}{a^2} \ , \tag{2.21}$$

which we can rewrite it as

$$1 - \Omega(a) = \frac{-k}{(aH)^2} \ , \tag{2.22}$$

where

$$\Omega(a) \equiv \frac{\rho(a)}{\rho_{\rm crit}} \quad and \quad \rho_{\rm crit}(a) \equiv 3H(a)^2. \tag{2.23}$$

Here, $\rho_{\rm crit}$ is the time- or scale-dependent *critical energy density* of the flat (Euclidean) Universe. As can be seen from Eqn. (2.23), $\Omega(a)$ parametrises the ratio of some energy density to the critical energy density. The critical density ratio is taken as $\Omega_{\rm crit} = 1$. Apparent from Eqn. (2.22), this value is in fact an unstable fixed point since the quantity $|\Omega - 1|$ diverges with time as comoving Hubble radius $(aH)^{-1}$ grows. This means that the only explanation for the observed near-flatness of the Universe today $\Omega(a_0) \sim 1$ is for the early Universe to be *extremely* flat (e.g. $|\Omega - 1| \le \mathcal{O}(10^{-16})$ at BBN and $|\Omega - 1| \le \mathcal{O}(10^{-55})$ at GUT).

## 2.3 The Inflation Primer

### 2.3.1 Inflation as a Solution

Both the horizon and the flatness problems arise due to the comoving Hubble radius, $(aH)^{-1}$, is increasing at all times. It is tempting to consider a simple solution by inverting its behaviour. In order to see this, it is important to understand what does the comoving Hubble radius mean and it's difference from the comoving time $\tau$ (or equivalently comoving horizon) in Eqn. (2.19). If two particles are separated by a distance *greater* than the comoving horizon, this means they have *never* been in contact. If they are separated by a distance greater than the comoving Hubble radius $(aH)^{-1}$, they are out of causal contact *today*. Two particles that were in causal contact before, may fall well out of contact, i.e. $\tau \gg (aH)^{-1}$ *now*, if the comoving Hubble radius in the early Universe was much larger than it is now. Hence we introduce a phase of decreasing Hubble radius. Since the Hubble parameter $H$ is approximately constant, scale factor $a$ *grows* exponentially during inflation resulting in the decrease of the comoving Hubble radius. This in turn fixes the flatness problem, i.e. for a non-flat universe the *decreasing* comoving Hubble radius drives the universe toward flatness

$$|\Omega(a) - 1| = \frac{1}{(aH)^2} \ , \tag{2.24}$$

where the solution $\Omega = 1$ is the attractor during inflation. Inflation similarly solves the horizon problem since decreasing comoving horizon means that the larger scales were smaller than the Hubble radius at early times and hence causally connected.

### 2.3.2 Basic Dynamics

From observing the Friedmann Equations, we arrive at two distinct phenomenological outcome, equivalent of our definition of decreasing comoving Hubble radius

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0, \tag{2.25}$$

namely, the *accelerated expansion* and a *negative pressure*. Accelerated expansion is a direct outcome of decreasing (or shrinking) comoving Hubble radius

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = \frac{-\ddot{a}}{(aH)^2} \Rightarrow \left\{\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0\right\}. \tag{2.26}$$

It is common to relate this expression the the first time derivative of the Hubble parameter

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon), \qquad \text{where} \qquad \epsilon := -\frac{\dot{H}}{H^2}, \tag{2.27}$$

with

$$\epsilon = -\frac{\mathrm{d}\ln H}{\mathrm{d}N} < 1, \tag{2.28}$$

where $\mathrm{d}N = \mathrm{d}\ln a$ measures the number of $e$-folds $N$ of inflationary expansion. Like the accelerated expansion, negative pressure is also complementary to decreasing Hubble radius. We understand that $\ddot{a} > 0$ requires negative pressure

$$p < -\frac{1}{3}\rho \ . \tag{2.29}$$

Finally, it is useful to note the significance of introducing inflation ($H \simeq$ const.) at the boundaries with respect to the evolution of the scale factor

$$a(\tau) = -\frac{1}{H\tau} \ , \tag{2.30}$$

where the $a = 0$ singularity is now pushed to infinite past $\tau_i \to -\infty$ and the scale factor *becomes* infinite at $\tau = 0$. This is since the Hubble parameter $H$ is assumed to be constant meaning inflation lasts forever with $\tau = 0$ being equivalent to the infinite future $t \to +\infty$. While this approximation obviously breaks down at times close to the end of inflation, it is still valid at early times. So in what follows, $\tau = 0$ no longer refers to the Big Bang, but instead the end of inflation. Hence by introducing inflation, we attain 'more time' before recombination. This results in apparently disconnected patches to be in causal contact.

### 2.3.3 Scalar field driving inflation

The simplest realisation for inflation is to consider a scalar field as an order parameter $\phi$, the *inflaton*, which parametrises the time-evolution of the inflationary energy density. The action for this scalar field (minimally) coupled to gravity,

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{2.31}$$

consists of an Einstein-Hilbert term $R$ and the scalar field action $S_\phi$ made of a canonical kinetic term and the potential $V(\phi)$ describing the self-interactions. The metric $g_{\mu\nu}$ is given at Eqn. (2.1) while the stress-energy tensor for the scalar field is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_\phi \tag{2.32}$$

where $\mathcal{L}_\phi$ is the Lagrangian of the action $S_\phi$. The equation of motion for the field can be calculated as

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + V_{,\phi} = 0 \ , \tag{2.33}$$

where $V_{,\phi} = \frac{\mathrm{d}V}{\mathrm{d}\phi}$. For the case of homogeneous field $\phi(t,\mathbf{x}) := \phi(t)$ and the stress-energy tensor satisfying perfect fluid, $T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$, with

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \ , \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \ , \end{aligned} \tag{2.34}$$

the equation of state becomes

$$\omega_\phi := \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \ . \tag{2.35}$$

This equation tells us that the scalar field can cause negative pressure ($\omega_\phi < 0$) and acceleration ($\omega_\phi < -\frac{1}{3}$) when potential energy of the scalar field $V(\phi)$ is sufficiently larger than the kinetic energy $\frac{1}{2}\dot{\phi}^2$. Finally, the Friedmann equations are solved to determine the dynamics of the scalar field and also the FRW background which are written as

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \qquad \text{and} \qquad H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \ . \tag{2.36}$$

### 2.3.4 Slow-roll inflation

The equation for an accelerating Universe, dominated by a homogeneous scalar field, was given in Eqn. (2.27). For the scalar field driving inflation, $\epsilon$ is called the *slow-roll parameter* and can be written also as

$$\epsilon := \frac{3}{2}(\omega_\phi + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \ . \tag{2.37}$$

Inflation occurs if $\epsilon < 1$ and de Sitter limit corresponds to $\epsilon \to 0$, where in de Sitter spacetime, the potential energy dominates over the kinetic energy $\dot{\phi}^2 \ll V(\phi)$. For inflation to last long enough to generate the primordial signatures we observe, the second time derivate of the field must also be small $|\ddot{\phi}| \ll |3H\dot{\phi}| \ , \ |V_{,\phi}|$. This introduces a second slow-roll parameter

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \frac{1}{2\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}N} \ , \tag{2.38}$$

where $|\eta| < 1$ makes sure that the change of $\epsilon$ per $e$-fold is small. In the slow-roll regime, the background evolution in Eqn. (2.36) further simplifies to

$$\dot{\phi} \simeq -\frac{V_{,\phi}}{3H} \qquad \text{and} \qquad H^2 = \frac{1}{3}V(\phi) \ , \tag{2.39}$$

and the spacetime is approximately de Sitter, i.e. $a(t) \sim \exp(Ht)$. Inflation ends when the kinetic energy becomes comparable to the potential energy, i.e. when the slow-roll conditions are violated $\epsilon(\phi_{end}) := 1$.

## 2.4 Cosmological Perturbations

In the previous sections we introduced basic concepts in understanding a simple homogenous Universe. We now begin our review of formulating the deviations from this simplicity.

### 2.4.1 Linear perturbations

Planck satellite and its predecessors consistently measured the Universe at the time of recombination (CMB) to be very homogeneous with inhomogeneities of order $\mathcal{O}(10^{-5})$. These inhomogeneities may then be naturally realised by linearly separating the cosmological quantities $X(t,\mathbf{x})$, e.g. $T_{\mu\nu}(t,\mathbf{x})$,

$g_{\mu\nu}(t,\mathbf{x})$, etc., into a homogeneous background $\bar{X}(t)$ and a perturbative term $\delta X(t,\mathbf{x})$. Scalar field and metric perturbations around the homogeneous background $\bar{\phi}(t)$ and $\bar{g}_{\mu\nu}(t)$ for inflation can be written as

$$\phi(t,\mathrm{x}) = \bar{\phi}(t) + \delta\phi(t,\mathrm{x}), \qquad \text{and} \qquad g_{\mu\nu}(t,\mathrm{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathrm{x}), \tag{2.40}$$

with spatially flat FRW metric,

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + 2aB_i\mathrm{d}x^i\mathrm{d}t + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]\mathrm{d}x^i\mathrm{d}x^j, \tag{2.41}$$

where $\Phi$ is the *lapse* (a 3-scalar), $B_i$ is the *shift* (a 3-vector), $\Psi$ is the spatial *curvature perturbation* (a 3-scalar) and $E_{ij}$ is a symmetric traceless spatial 3-tensor called *shear*. The metric perturbations are coupled to *matter perturbations* during inflation since the inflaton is the dominant contributor to the stress-energy of the Universe. The perturbations to the stress energy tensor is then given as

$$\begin{aligned} T^0_0 &= -(\bar{\rho} + \delta\rho) \\ T^0_i &= (\bar{\rho} + \bar{p})av_i \\ T^i_0 &= -(\bar{\rho} + \bar{p})(v^i - B^i)/a \\ T^i_j &= \delta^i_j(\bar{p} + \delta p) + \Sigma^i_j \ . \end{aligned} \tag{2.42}$$

where $\Sigma^i_j$ is called the *anisotropic stress*. However the realisations in Eqns. (2.41) and (2.42) are not unique, they depend on the choice of the coordinates or the *gauge choice*.

### 2.4.2 Gauge invariance

Gauge choice determines how the background maps onto physical spacetime with perturbation. Gauge choice is ubiquitous to all areas of physics and carry much significance. An inappropriate choice of gauge may result in some non-physical perturbations popping-up, or some *real* perturbations vanish entirely. To avoid such issues, one must study the gauge-invariant combinations of perturbations which by definition *cannot* be removed by coordinate transformations. Two such parameters in cosmology are the *curvature perturbation* on *uniform-density hyper surfaces*

$$\zeta \equiv \Psi + \frac{H}{\dot{\rho}}\delta\rho \ , \tag{2.43}$$

and *comoving curvature perturbation*

$$\mathcal{R} = \Psi - \frac{H}{\bar{\rho} + \bar{p}}\delta q \ , \tag{2.44}$$

where parameter $\delta q$ is the momentum density $(\delta q)_{,i} \coloneqq (\bar{\rho} + \bar{p})v_i$. The parameters $\zeta$ and $\mathcal{R}$ remain constant on superhorizon scales and their autocorrelations are equal to each other at horizon crossing, making them essential for cosmological calculations. In the next section we will give definitions for various statistical terms often used in cosmology.

### 2.4.3 Basic statistics

The central statistical object in cosmology in the power spectrum of primordial *scalar* fluctuations $\mathcal{R}$. This is shown with by taking an expectation value of the fluctuations in the form

$$\langle\ \mathcal{R}_{\mathbf{k}}\mathcal{R}_{\mathbf{k}'}\rangle = (2\pi)^3\delta(\mathbf{k} + \mathbf{k}')\ P_{\mathcal{R}}(k)\ , \tag{2.45}$$

or in the scale-independent representation

$$\Delta^2_s \coloneqq \Delta^2_{\mathcal{R}} = \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k)\ . \tag{2.46}$$

The scale independence is parametrised by the scalar spectral index

$$n_s - 1 := \frac{d\ln\Delta_s^2}{d\ln k} \ , \tag{2.47}$$

where for scale invariant spectrum $n_s = 1$. The *running* of the spectral index is defined as

$$\alpha_s := \frac{dn_s}{d\ln k} \ , \tag{2.48}$$

with which the power spectrum is approximated as

$$\Delta_s^2(k) = A_s(k_\star)\left(\frac{k}{k_\star}\right)^{n_s(k_\star)-1+\frac{1}{2}\alpha_s(k_\star)\ln(k/k_\star)} \tag{2.49}$$

where $k_\star$ is some arbitrary scale.

### 2.4.4 ADM action

Although it may not seem very simple at the first glance, Arnowitt-Deser-Misner (ADM) [7] formalism has much importance in modern methods in cosmology as it lets one to explicitly separate temporal and spatial degrees of freedom. This allows construction of the interaction Hamiltonian $H_I$.[2] The ADM formalism is defined as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t) \ , \tag{2.50}$$

where the spacetime is sliced into three dimensional hyper surfaces and $g_{ij}$ behaves as a three dimensional metric on constant time slices. In this formalism we see the change in variables *lapse* $\Phi \to N(\mathbf{x})$ and *shift* $B_i \to N_i(\mathbf{x})$. While $N(\mathbf{x})$ and $N_i(\mathbf{x})$ describe the same physics, one major difference is that they are simply algebraic Lagrange multiplies in the scalar action in Eqn. (2.31)

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[NR^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot\phi - N^i\partial_i\phi)^2 - Ng^{ij}\partial_i\phi\partial_j\phi - 2V\right] \tag{2.51}$$

and (defining extrinsic curvature as $K_{ij} = N^{-1}E_{ij}$ ) ,

$$E_{ij} = \frac{1}{2}(\dot g_{ij} - \nabla_i N_j - \nabla_j N_i) = NK_{ij} \quad \text{and} \quad E = E^i_i = E_{ij}g^{ij} \ . \tag{2.52}$$

Although we will not be calculating cosmological perturbations in this paper, it is worth pointing out the algebraic solutions for the parameters $N$ and $N_i$. Namely, since these parameters are Lagrange multiplies, we can write their equation of motion (or *constaint* equations) simply as

$$\begin{gathered} R^{(3)} - 2V - g^{ij}\partial_i\phi\partial_j\phi - N^{-2}\left[E_{ij}E^{ij} - E^2 + (\dot\phi - N^i\partial_i\phi)^2\right] = 0 \ , \\ \nabla_i\left[N^{-1}(E^i_j - E\delta^i_j)\right] = 0 \ . \end{gathered} \tag{2.53}$$

Plugging the solutions for these parameters back into the action leaves $g_{ij}$ and $\phi$ as dynamical variables.

### 2.4.5 Comoving gauge

The comoving gauge is one of the more popular gauge choices and will be central various calculations we will make in the following sections. We begin by choosing

$$\delta\phi = 0 \ , \tag{2.54}$$

[2] See [8] for a recent re-print of the original work by the authors.

and (for ADM formalism)

$$g_{ij} = a^2 \left[(1 - 2\mathcal{R})\delta_{ij} + h_{ij}\right] \quad \text{where} \quad \partial_i h_{ij} = h^i_i = 0 \ . \tag{2.55}$$

It is not uncommon in cosmology for the parameters $\zeta$ and $\mathcal{R}$ to be mistakenly written for one another. We will try avoid this by being explicit on our definition and follow Eqn. (2.44) and Eqn. (2.43). In this gauge, inflaton is unperturbed where all scalar degrees of freedom area *eaten*[3] by the metric fluctuation $\mathcal{R}(t, \mathbf{x})$. We will complete this section by giving the results for curvature perturbation in comoving gauge. If one re-writes the curvature perturbation in Eqn. (2.55) as

$$g_{ij} = a^2 e^{2\mathcal{R}} \delta_{ij} \ , \tag{2.56}$$

solving the constraint equations in Eqn. (2.55) for the Lagrange multiplies $N = 1 + \delta N$ and $N_i = \partial_i \chi$ , one finds

$$\delta N = \frac{\dot{\mathcal{R}}}{H} \quad \text{and} \quad \nabla^2 \chi = \epsilon \, \dot{\mathcal{R}} - \frac{\nabla^2 \mathcal{R}}{H} \ . \tag{2.57}$$

where $\epsilon$ is the slow roll parameter introduced in Section §2.3.4. We will return to these results later when introducing the so called *in-in* formalism. Next, we will introduce the effective field theory of microscopic physics of inflation. Effective field theory approach allows for the construction of a very general representation and arguably more intuitive in considering links to condensed matter theories.

## 2.5 Effective field theory of inflation

We pedagogically introduced the usual approach to inflation in the preceding sections. There, we started by introducing a Lagrangian for a scalar field $\phi$ in Eqn. (2.31) and calculated the equation of motion for $\phi$ along with the Friedman equations for the FRW metric, i.e. in Eqn. (2.36). We then introduced perturbations of the scalar field in Eqn. (2.40), and the FRW metric in Eqn. (2.41). The standard approach in obtaining inflationary observables is then via calculating the action for these fluctuations, where solving these perturbed equations depends on the *a priori* assumptions about the microscopic physics sourcing the background. Effective field theory (EFT) of inflation [9–11] provides an alternative approach and also allows a more generalised formalism. In EFT of inflation, one takes the accelerating spacetime, $|\dot{H}| \ll H^2$, with quasi-de Sitter background $H(t)$ as *given* and writes the *effective* action for the fluctuations *directly*. Effective field theories are in general much simpler than a UV complete theories and they are also broadly applicable to many scenarios. Moreover for our purposes, EFT of inflation provides perhaps a most natural formalism to justify making direct analogies with condensed matter physics due to the gauge invariance of fluctuations and decoupling from gravity. We will explore this issue later in Section §3.4.2. In what follows, we will introduce the EFT of inflation.

### 2.5.1 Unitary gauge

We begin with the observation that inflation breaks time diffeomorphisms[4] by coming to an end. This is realised by considering quasi-de Sitter background having a preferred spatial slicing. The preferred slicing can be given by a function $\tilde{t}(x)$ (with time-like gradient) which non-linearly realises time diffeomorphisms. We can consider this slicing to be given by a time evolving scalar $\phi(t)$, a *physical clock*, which allows the natural ending of inflation. With this observation, one then determines the most general Lagrangian in the *unitary gauge*. Here, the unitary gauge is the one in which the coordinate $t$ is chosen such that the surfaces of constant $\tilde{t}$ are also a constant value of the scalar $\phi$, i.e.

$$\delta\phi(t, \mathbf{x}) = 0 \ , \tag{2.58}$$

[3] By *eaten*, we follow the common jargon (see e.g. [9]) in which we refer to the fact that the degrees of freedom are *parametrised* by (in this case) $\mathcal{R}$.

[4] Here and throughout when discussing EFT of inflation, we will refer to transformations as diffeomorphisms to match the vocabulary used by EFT of inflation theorists.

where time diffeomorphisms are fixed and the gauge symmetry is now spontaneously broken.[5] The choice in Eqn. (2.58) means that there are no scalar perturbations but only metric fluctuations, i.e. the scalar degree of freedom is *eaten* by the metric.

### 2.5.2 The most general Lagrangian

The most general unitary gauge Lagrangian is composed of all the diffeomorphism invariant terms. In the last section, we discussed the unitary gauge choice where slicing $\tilde{t}$ coincide with coordinate $t$. This means the slicing degree of freedom, $\tilde{t}$, does not appear directly in the action. Moreover, any general function of $\tilde{t}$, becomes a function of time, $f(t)$, and can be used freely in the action. Finally in the unitary gauge, the gradient of $\tilde{t}$ becomes a delta function

$$\partial_\mu \tilde{t} = \delta^0_\mu \ , \tag{2.59}$$

which results in every *free* tensor attaining an upper 0 index (e.g. $g^{00}$, $R^{00}$). In addition to such terms, the *extrinsic curvature*, $K_{\mu\nu}$, is also allowed. Extrinsic curvature can be calculated by defining a *unit* vector *perpendicular* to constant $\tilde{t}$ surfaces,

$$n_\mu = \frac{\partial_\mu \tilde{t}}{\sqrt{-g^{\mu\nu}\partial_\mu \tilde{t} \partial_\nu \tilde{t}}} \ , \tag{2.60}$$

where the *induced spatial metric* on constant $\tilde{t}$ surfaces is given as

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \ , \tag{2.61}$$

which is used to project any tensor on the constant $\tilde{t}$ surfaces. The covariant derivative of $n_\mu$ projected on these surfaces gives the extrinsic curvature

$$K_{\mu\nu} \equiv h^\sigma_\mu \nabla_\sigma n_\nu \ . \tag{2.62}$$

Lastly, in addition to these terms, the Riemann tensor $R_{\mu\nu\rho\sigma}$ and its covariant derivates, along with their polynomials, also go in the Lagrangian as they are invariant under *any* diffeomorphism. The most general Lagrangian is then written as [9]:

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\,\Big[&\frac{1}{2}M_{pl}^2 R - c(t)g^{00} - \Lambda(t) + \frac{1}{2!}M_2(t)^4(g^{00}+1)^2 + \dots \\ &+ \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 - (\frac{\bar{M}_1(t)^3}{2})(g^{00}+1)\delta K^\mu_\mu \\ &- \frac{\bar{M}_2(t)^2}{2}(\delta K^\mu_\mu)^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots\Big] \end{aligned} \tag{2.63}$$

where $\delta K_{\mu\nu}$ is the variation of the extrinsic curvature $\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H h_{\mu\nu}$. The linear terms are fixed by the *unperturbed* FRW background evolution. Notice that only the these terms contribute to the stress tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \ , \tag{2.64}$$

where Friedmann equations for the flat FRW universe are

$$H^2 = \frac{1}{3M_{pl}^2}[c(t) + \Lambda(t)] \quad \text{and} \quad \dot{H} + H^2 = -\frac{1}{3M_{pl}^2}[2c(t) - \Lambda(t)] \ . \tag{2.65}$$

[5] Note that it is only the time diffeomorphisms that are broken. The action is still symmetric in spatial diffeomorphisms.

Solving these equations for $c(t)$ and $\Lambda(t)$ give

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\, \Big[ & \frac{1}{2}M_{pl}^2 R + M_{pl}^2 \dot{H} g^{00} - M_{pl}^2(3H^2 + \dot{H}) + \frac{1}{2!}M_2(t)^4(g^{00}+1)^2 \\ & + \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 - \frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu_\mu \\ & - \frac{\bar{M}_2(t)^2}{2}(\delta K^\mu_\mu)^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots \Big] . \end{aligned} \tag{2.66}$$

Notice that the first two terms can be written to take a form similar to slow-roll with potential $V(\bar{\phi}) = M_{pl}^2(3H^2 + \dot{H})$ and the kinetic term $\dot{\bar{\phi}}^2 = -2M_{pl}^2\dot{H}$. Next, we will reintroduce the gauge invariance by introducing a Goldstone boson.

### 2.5.3 Introducing the Goldstone boson

The procedure of introducing the Goldstone boson follows making a gauge transformation to restore the invariance of the unitary gauge action for a non-Abelian gauge group [12]. We will not give a review of this topic here (see however Section §3.4.2), but directly apply the ideas to the unitary Lagrangian following mainly [9]. We start by focusing on the two linear terms in Eqn. (2.66) of the form

$$\int \mathrm{d}^4x\sqrt{-g}\,\left[A(t) + B(t)g^{00}(x)\right] . \tag{2.67}$$

From Eqn. (2.59), we see that the broken time diffeomorphisms can be shown as

$$t \to \tilde{t} = t + \xi^0(x) , \tag{2.68}$$

while the spatial diffeomorphisms are maintained,

$$\mathbf{x} \to \tilde{\mathbf{x}} = \mathbf{x} . \tag{2.69}$$

As a result, $g^{00}$ transforms as

$$g^{00}(x) \to \tilde{g}^{00}(\tilde{x}(x)) = \frac{\partial \tilde{x}^0}{\partial x^\mu}\frac{\partial \tilde{x}^0(x)}{\partial x^\nu} g^{\mu\nu}(x). \tag{2.70}$$

Next, we change the integration variable $x$ to $\tilde{x}$ and write the terms as

$$\tilde{x}^0(x) = \tilde{t} - \xi^0(\tilde{x}(x)) , \tag{2.71}$$

to get

$$\int \mathrm{d}^4\tilde{x}\sqrt{-\tilde{g}(\tilde{x})}\left[A(\tilde{t} - \xi^0 x(\tilde{x})) + B(\tilde{t} - \xi^0 x(\tilde{x}))\frac{\partial(\tilde{t} - \xi^0(x(\tilde{x})))}{\partial \tilde{x}^\mu}\frac{\partial(\tilde{t} - \xi^0(x(\tilde{x})))}{\partial \tilde{x}^\nu}\tilde{g}^{\mu\nu}(\tilde{x})\right] \tag{2.72}$$

Next, we define the Goldstone boson which transforms as a *scalar field* and an additional *shift* under time diffeomorphisms

$$\pi \to \pi = \pi - \xi^0(t, \mathbf{x}) . \tag{2.73}$$

Notice under the reparametrization $t \to t + \xi^0(t, \mathbf{x})$, the term $t + \pi$ is invariant. Finally for every $\xi^0$ we substitute the Goldstone mode

$$\xi^0(x(\tilde{x})) \to -\tilde{\pi}(\tilde{x}) . \tag{2.74}$$

Action then becomes

$$\int \mathrm{d}^4\sqrt{-g(x)}\left[A(t + \pi(x)) + B(t + \pi(x))\frac{\partial(t + \pi(x))}{\partial x^\mu}\frac{\partial(t + \pi(x))}{\partial x^\nu}g^{\mu\nu}(x)\right] , \tag{2.75}$$

where we dropped the tildes for simplicity. Generalising this approach for the action in Eqn. (2.66), one arrives at the general form of the unitary gauge action with the Goldstone boson [9]

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\,\Big[&\frac{1}{2}M_{pl}^2 R - M_{pl}^2(3H^2(t+\pi)+\dot{H})(t+\pi) \\ &+ M_{pl}^2\dot{H}(t+\pi)((t+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi) \\ &+ \frac{M_2(t+\pi)^4}{2!}((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi + 1)^2 + \dots\Big]\,, \end{aligned} \tag{2.76}$$

where we only included up to terms in the first line of Eqn. (2.66). This action is complicated! It involves many complicated interaction terms and also couplings between the gravity and the Goldstone boson. Nevertheless, one must take into account that Eqn. (2.76) represents the *most general unitary gauge action*. Moreover, it greatly simplifies at the relevant energy scale, namely at the *decoupling limit*.

### 2.5.4 Decoupling Limit

One of the main reasons for the EFT formalism of inflation to be attractive is that the physics of the Goldstone boson *decouples* from the graviton at sufficiently high energies. This reasoning follows the non-Abelian gauge theories and gapless-ness of Goldstone boson [12, 13], also see Section §3.4.2. In the EFT of inflation formalism, the mixing between the longitudinal (e.g. Golstone) and the transverse (e.g. graviton) components of the gauge field becomes irrelevant. This can be realised by observing that the leading order contribution to the mixing in Eqn. (2.76) comes from the term

$$\mathcal{L}_{\text{mix}} \sim M_{\text{pl}}^2\dot{H}\dot{\pi}\delta g^{00}\,, \tag{2.77}$$

which by normalisation $\pi_c \sim M_{\text{pl}}|\dot{H}|^{1/2}\pi$ and $\delta g_c^{00} \sim M_{\text{pl}}\delta g^{00}$ takes the form

$$\mathcal{L}_{\text{mix}} \sim \mathcal{E}_{\text{mix}}\pi_c\delta g_c^{00} \quad \text{where} \quad \mathcal{E}_{\text{mix}} \sim \epsilon^{1/2}H\,, \tag{2.78}$$

The variable $\epsilon$ is the slow roll parameter in Eqn. (2.27). This means that in the regime $E \gg \mathcal{E}_{\text{mix}}$, *only* Goldstone boson controls the dynamics. We realise this *decoupling* limit by taking $M_{\text{pl}} \to \infty$ and $\dot{H} \to 0$ where we hold $M_{\text{pl}}^2\dot{H} \gg H^4$ fixed. This allows one to ignore the metric perturbations and use only the (unperturbed) background de Sitter metric $\bar{g}^{\mu\nu}$. Recall the transformation for the gauge invariant parameter $g^{00}$ in Eqn. (2.70). At the decoupling limit, this becomes

$$\boxed{g^{00} \to \partial_\mu(t+\pi)\partial_\nu(t+\pi)\bar{g}^{\mu\nu} = -1 - 2\dot{\pi} - \dot{\pi}^2 + \frac{(\partial_i\pi)^2}{a^2}\,,} \tag{2.79}$$

where the Lagrangian for (gauge invariant) fluctuations simplifies considerably,

$$\boxed{S_\pi = \int \mathrm{d}^4x\sqrt{-g}\,\Big[\,M_{\text{pl}}^2\dot{H}\big(-\dot{\pi}^2 + \frac{(\partial_i\pi)^2}{a^2}\big) + 2M_2^4\big(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\big) + \dots\Big]\,.} \tag{2.80}$$

We see above that at the quadratic level, the coefficient of the spatial kinetic term $(\partial_i\pi)^2$ is completely fixed by the background. This is not the case for the time kinetic term $\dot{\pi}^2$ as it receives an extra contribution with coefficient $\sim M_2^4$. In order to maintain a *stable action* we must have $(-M_{\text{pl}}^2\dot{H} + 2M_2^4)\,\dot{\pi}^2 > 0$. An object of interest for various calculations is the *speed of sound* $c_s$.

$$c_s^{-2} = 1 - \frac{2M_2^4}{M_{\text{pl}}^2\dot{H}}\,, \tag{2.81}$$

which may be different than one, i.e. $c_s \neq 1$, even though we are referring to a massless Goldstone boson. This is because the time and spatial terms have different coefficients.

### 2.5.5 The $\mathcal{R} \to \pi$ correspondance

We will end our review of EFT of inflation by pointing out the direct correspondence between the gauge invariant curvature perturbation $\mathcal{R}$ and the Goldstone boson $\pi$. This can most easily be observed in the comoving gauge which is introduced in Section §2.4.5. In EFT formalism, comoving gauge is equivalent to unitary gauge since $\pi \sim \delta\phi$, where it is realised by choosing $\pi = 0$ and defining the spatial metric as in Eqn. (2.55). Since the metric is unperturbed in the decoupling limit, the choice $\pi = 0$ amounts to a time diffeomorphism

$$t \to t - \pi(t, \mathbf{x}) \;, \tag{2.82}$$

which gives

$$\boxed{\mathcal{R}(t, \mathbf{x}) = -H\pi(t, \mathbf{x}) \;.} \tag{2.83}$$

This completes our review of the EFT of inflation. In the next section, we will begin reviewing the observational aspects of early Universe cosmology.

## 2.6 Observations of the Early Universe

In this section, we will review the correspondence of early Universe physics with the CMB signal and introduce statistical methods for these calculations. In the next section we will discuss the state-of-the-art non-Gaussianity calculations both in experiments and also in theory.

### 2.6.1 The cosmic microwave background

We have mentioned the cosmic microwave background (CMB) radiation few times before without going into much detail on its properties. The CMB is a ‘relic’ black body radiation which dates back to when the Universe was 380,000 years old ($z \simeq 1100$) when the electrons first became bounded by protons forming atoms (i.e. recombination), allowing photons to freely propagate for the first time. The most recent detection of the CMB was carried out by Planck satellite (see latest results from 2015 [1]). The CMB signal is particularly valuable for understanding the physics of the early Universe as it can be isolated to include only the primordially sourced density fluctuations. This is essentially due to two reasons. First, the thermodynamics effects that influence the CMB signal are well understood between safely after inflation and recombination are well understood. Second, the *background cosmology* that effects the propagating photons from recombination to the detectors *today* is also well understood. These effects can be accounted for at the observed CMB signal by calculating the relevant transfer functions and projection effects. What remains after this *deconvolution* is the *primordial fluctuations* from the very early Universe (see Figure 2.1). In the previous sections we introduced the primordial comoving curvature perturbation $\mathcal{R}$ and discussed its relevance to cosmological observations. Not only that this quantity is *independent* of how we choose our coordinates describing the fluctuations, but it’s amplitude is also *frozen* at the horizon crossing during inflation. Since these *superhorizon* modes enter *back* into the horizon safely[6] after the end of inflation, we can relate the fluctuations of $\mathcal{R}$ to the measured temperature fluctuations by simply accounting for the physics between the horizon re-entry and CMB recombination (see Figure 2.2) in the form

$$T_{\rm cmb}(\tau) = \Delta^{\mathcal{T}}(k, \tau, \tau_\star)\mathcal{R}_{\mathbf{k}}(\tau_\star) \;, \tag{2.84}$$

where $T_{\rm cmb}(\tau)$ is the measured value at later time $\tau$. The $\Delta^{\mathcal{T}}$ is the *transfer function* between the horizon re-entry for $\mathcal{R}$ fluctuations at time $\tau_\star$ and the observation at $\tau$.

[6]Modes that cross the horizon during inflation enter back not immediately after the end of inflation but a while later, not being affected by reheating, which is a much lesser known and highly non-linear phenomena.

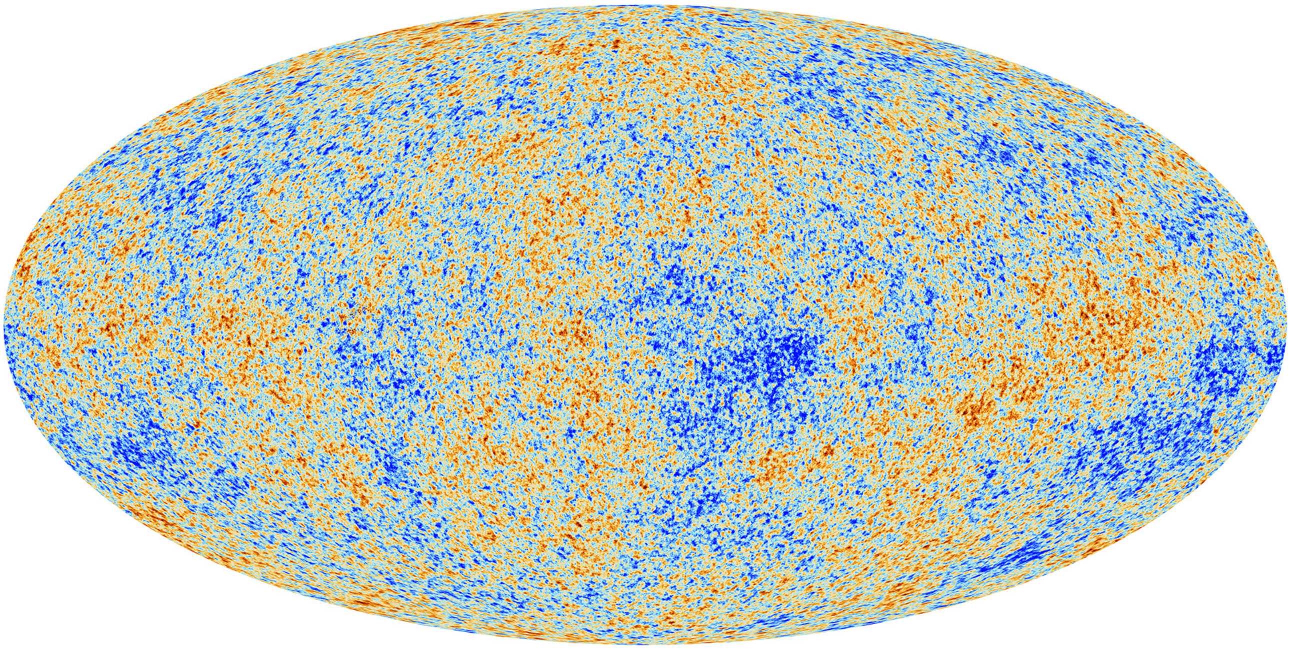


Figure 2.1: [Image Credit: Planck]. The CMB observed by Planck satellite. Image shows that anisotropies in temperature fluctuations with blue (red) areas represent directions on the sky where the temperature of the CMB is $\mathcal{O}(10^{-5})$ below (above) the mean temperature value ($\bar{T} = 2.7K$). The temperature fluctuations $\Delta T$ correspond to the (primordial) density variations since photons loose energy while *climbing* up out of gravitational potentials at the *overdense* (i.e. blue) regions. The properties of the physics of the early Universe, as well as the evolution of the cosmological background, manifests in the statistical characteristics of the CMB signal.

### 2.6.2 Temperature Fluctuations and The Power Spectrum

Following the arguments in the previous section, we are now interested in calculating the power spectrum $P_{\mathcal{R}}(k)$ for curvature perturbations. Inflation predicts

$$k^3 P_{\mathcal{R}}(k) \propto k^{n_s - 1} \ , \tag{2.85}$$

where $n_s \simeq 1$ and

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k) \ . \tag{2.86}$$

The general function that relates the primordial fluctuations to the measurements consists of a transfer function and a projection as we explained before. Since the CMB signal is very homogenous, the calculations are most easily done in phase space with spherical harmonic expansion. The transfer function becomes

$$\Delta_l^{\mathcal{T}}(k) = \int_0^{\tau_0} \mathrm{d}\tau \ \underbrace{S^{\mathcal{T}}(k,\tau)}_{\text{Source Terms}} \ \underbrace{P_l^{\mathcal{T}}(k[\tau_0 - \tau])}_{\text{Bessel Functions}} \ , \tag{2.87}$$

with the multipole moments $l$ and $\tau_0$ is some later time. The transfer function is in general calculated numerically using Boltzmann codes CMBFAST or CAMB. The *angular* power spectrum of the CMB *temperature* measurements then become

$$C_l^{TT} = \frac{2}{\pi} \int k^2 \ \mathrm{d}k \ P_{\mathcal{R}}(k) \left(\Delta_l^{\mathcal{T}}(k)\right)^2 \ . \tag{2.88}$$

The angular power spectrum is a very widely used statistical tool which contains most of the information CMB map holds in a very compact way. The angular spectrum of CMB temperature fluctuations published by Planck collaboration is given in Figure 2.3.

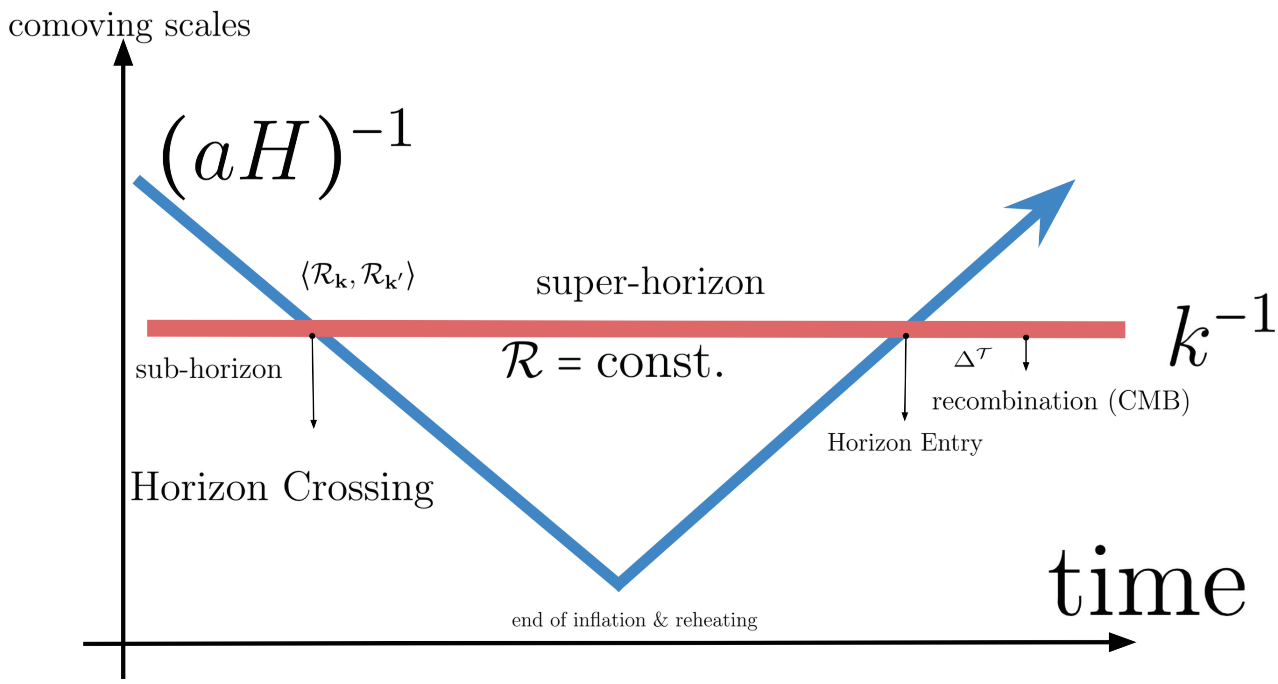



Figure 2.2: Chronology of fluctuations in the early Universe. Red line shows the comoving scales $k^{-1}$ that remains constant throughout the evolution. Blue line represents the comoving Hubble radius $(aH)^{-1}$ which shrinks during inflation but increases once inflation has ended. Fluctuations are created on sub-horizon scales quantum mechanically. They remain constant once the modes exit the horizon and remain frozen until long after the end of inflation, being unaffected by the influence of the reheating process. From horizon re-entry to the CMB recombination, the background effects can be realised by calculating the transfer functions $\Delta^{\mathcal{T}}$ based on thermodynamical principles (see also [6]).

## 2.7 Non-gaussiantities

In the case the primordial fluctuations of $\mathcal{R}$ are purely Gaussian, power spectrum will contain *all* the statistical information available. However, deviation from Gaussianity can be caused by a number of things including a wide range of late time effects[7]. What is of great interest for the early Universe cosmologists, however, is the *primordially* sourced non-Gaussianity. Primordial non-Gaussianity can be produced by the quantum effects of microphysics of inflation, or by the non-linear classical evolution of super-horizon modes (see e.g. for qualitative review [14–16]).

### 2.7.1 Primordial bispectrum

While there is one way for a distribution to be Gaussian, there are *infinitely* many ways for that distribution to deviate from perfect Gaussianity. One reasonable way to calculate these deviations, however, is Taylor expanding to probability distribution around a Gaussian one. In this representation, the leading effect comes from the Fourier transform of the three point function and called the *bispectrum*

$$\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \mathcal{B}_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \ . \tag{2.89}$$

The translation invariance of the background introduces the delta function (momentum conservation), rotational invariance reduces the independent parameters to two which characterise the *shape* of the bispectrum (e.g. $k_2/k_1$, $k_3/k_1$).

[7]For the CMB signal these include the *foreground* (i.e. Galactic and extra-Galactic sources), *lensing* effects generated after recombination and non-linearities in the transfer function $\Delta_l^{\mathcal{T}}$, relating curvature perturbation at the horizon re-entry $\mathcal{R}$ to temperature anisotropies at the CMB, $\Delta T$. As we discussed before, these effects can be removed from the signal since the physics that sources them are relatively well understood.

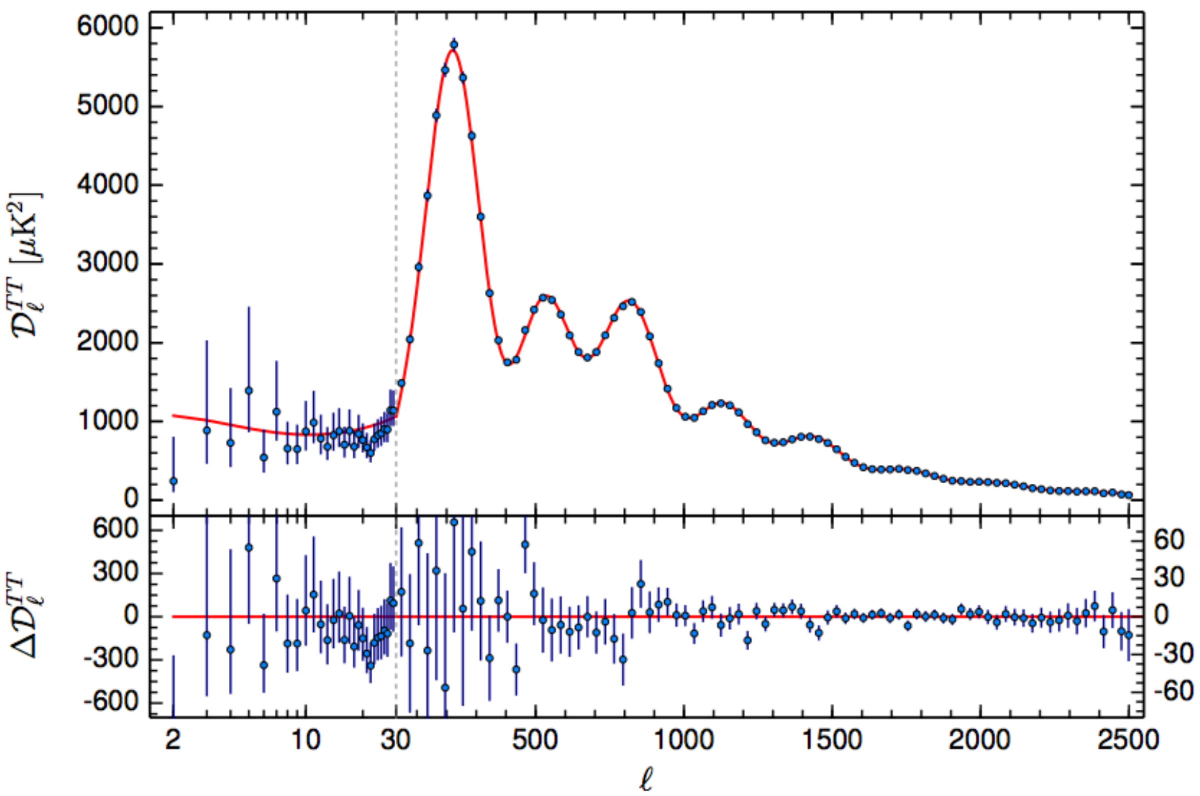



Figure 2.3: [`Image Credit: Planck 2015`]. Angular power spectrum calculated by the Planck satellite with $\mathcal{D}_l^{\mathcal{TT}} = l(l+1)\, C_l^{\mathcal{TT}}/(2\pi)$ where $l$ is the multipole moment. The $C_l^{\mathcal{TT}}$'s are directly related with the curvature perturbation $P_{\mathcal{R}}(k)$. The power spectrum tells us that the theoretical calculations match very well with the data and we understand the physics that shape the CMB power spectrum. Moreover, it proves that the Fourier modes of primordial fluctuations are of the *same phase*. Which is a direct consequence of inflation! (see e.g. for discussion on topic [5]).

### 2.7.2 Shapes of bispectrum

Bispectrum contains a lot of information about the source of the non-Gaussianity, allowing to distinguish between models. Perhaps simplest way to calculate non-Gaussianity is by introducing a non-linear correction to the otherwise Gaussian curvature perturbation $\mathcal{R}_g$,

$$\mathcal{R}(x) = \mathcal{R}_g(x) - \frac{3}{5} f_{NL}^{local} (\mathcal{R}_g^2 - \langle\mathcal{R}_g\rangle^2)\ . \tag{2.90}$$

This correction is called *local non-Gaussianity* due to it being local in real space. By combining Eqns. (2.90) and (2.89) one calculates the bispectrum of the local non-Gaussianity as

$$\mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3) = \frac{6}{5} f_{NL}^{local} \times \underbrace{[P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_2) + P_{\mathcal{R}}(k_2)P_{\mathcal{R}}(k_3) + P_{\mathcal{R}}(k_3)P_{\mathcal{R}}(k_1)]}_{(\star)\quad P_{\mathcal{R}}(k)\propto Ak^{-3}}\ , \tag{2.91}$$

where we apply the scale-invariance $(\star)$ to get

$$\mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3) = \frac{6}{5} f_{NL}^{local} \times A^2 \left[\frac{1}{(k_1k_2)^3} + \frac{1}{(k_2k_3)^3} + \frac{1}{(k_3k_1)^3}\right]\ . \tag{2.92}$$

Consider the ordering $k_3 \leq k_2 \leq k_1$, which can be assumed without loss of generality. In this case, the bispectrum is largest when $k_3 \ll k_1 \sim k_2$. This is called the *squeezed limit* and the bispectrum for local non-Gaussianity becomes

$$\lim_{k_3 \ll k_1 \sim k_2} \mathcal{B}_{\mathcal{R}}(k_1, k_2, k_3) = \frac{12}{5} f_{NL}^{local} \times P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_3) \ . \tag{2.93}$$

In general, the shape of non-Gaussianity is a quite important measure on the properties of primordial perturbations. Bispectrum can be written as

$$B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{\mathcal{S}(k_1, k_2, k_3)}{(k_1 k_2 k_3)^2} \cdot \Delta^2_{\mathcal{R}}(k_\star) \ , \tag{2.94}$$

where $\Delta^2_{\mathcal{R}}(k_\star) = k_\star^3 P_{\mathcal{R}}(k_\star)$ . Some of the other shapes include *equilateral* non-Gaussianity, where the shape peaks once the configuration is equilateral $k_1 = k_2 = k_3$ or *orthogonal* non-Gaussianity where the observed shape is orthogonal to both equilateral and local templates $\mathcal{S}_{\text{ortho.}} \cdot \mathcal{S}_{\text{equil.}} = \mathcal{S}_{\text{ortho.}} \cdot \mathcal{S}_{\text{loc.}} := 0$ .

## 2.8 Cosmological quantum corrections

In this section we will introduce one of the tools available for calculating cosmological corrections to what otherwise would be homogenous and Gaussian distributions. Namely, we will introduce the *in-in* formalism for calculating the corrections arise from effects of micro-scale physics by using quantum field theory methods. In the following chapters we will return to this method when calculating the effects of the disorder phenomena we will soon introduce.

### 2.8.1 *In-In* Formalism: Corrections from quantum effects

The origin of the *in-in* method dates back to J. Schwinger [17] and L. Keldysh [18] and involves calculating closed-time path (CTP) integrals. This method was then studied and formulated for cosmology in order to calculate perturbative quantum-mechanical effects in the early Universe by J. Maldacena [19] and S. Weinberg [20]. We will review the contemporary cosmological formalism in what follows.

#### Review

When calculating correlation functions in particle physics quantum field theories the essential object is the $S$-matrix which describes transition probabilities from a state very early in time to a state vary far in future

$$\langle out | \, S \, | in \rangle = \langle out(+\infty) | in(-\infty) \rangle \ . \tag{2.95}$$

In cosmology, on the other hand, we are not necessarily interested in calculating transition probabilities with $S$-matrix elements but instead calculating various expectation values at a *fixed time.* Hence the conditions are imposed on the fields only at very early times[8]. The expectation value takes the form

$$\langle Q \rangle = \langle in | \, Q(t) \, | in \rangle \ , \tag{2.96}$$

where $|in\rangle$ is the vacuum with interacting degrees of freedom at a very early time $t_0$ and $t$ is some time later like the *horizon crossing* where we wish to calculate our $n$-point functions of the parameters we are interested with. In Figure 2.4 we demonstrate the path integral given in Eqn. (2.96). The point of return represents where the expectation values are calculated. We now present the 'master formula' of the *in-in* formalism (see also the discussion in Section §4.3.2),

$$\langle Q(t) \rangle = \langle 0 | \left[ \bar{\mathrm{T}} \exp\left( i \int_{-\infty(1-i\epsilon)}^{t} \mathrm{d}t' H_I(t') \right) \right] Q^I(t) \left[ \mathrm{T} \exp\left( -i \int_{-\infty(1-i\epsilon)}^{t} \mathrm{d}t'' H_I(t'') \right) \right] |0\rangle \ , \tag{2.97}$$

[8] This results in the same form interaction picture fields due to Equivalence Principle, see [20] .


Figure 2.4: Graphical demonstration of the path integral in the *in-in* formalism.

where $\bar{\mathrm{T}}$ is the *anti*-time ordering symbol and $i\epsilon$ used to project interacting vacuum state $|in\rangle$ onto free vacuum state. This expression can then be *perturbatively* expanded to give more convenient equivalent form, which we will also use later in the work

$$\langle Q(t)\rangle = \sum_{n=0} i^n \int_{-\infty(1-i\epsilon)}^{t} \mathrm{d}t_n \int_{-\infty(1-i\epsilon)}^{t_n} \mathrm{d}t_{n-1} \dots \int_{-\infty(1-i\epsilon)}^{t_2} \mathrm{d}t_1 \\ \times \left\langle \left[H_I(t_1), \left[H_I(t_2), \dots \left[H_I(t_n), Q^I(t)\right] \dots \right]\right]\right\rangle \ . \tag{2.98}$$

# Chapter 3

# Condensed Matter Phenomena & Disorder

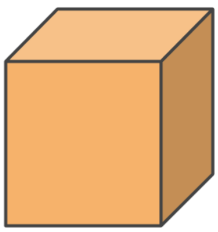
In this chapter we will discuss the analogies between condensed matter physics and early Universe cosmology. Our aim is to give a clear description of the condensed matter phenomena that can be related to cosmological scenarios. We believe when making these analogies, it is most appropriate to start by considering the notion of 'complexity' which we introduce in Section §3.1. Then, starting from Section §3.2, we focus mainly on the mechanism of *disorder*.

## 3.1 Complexity and emergence

In the first chapter, we suggested the idea of complexity as an inherent characteristic of the underlying physics in the early Universe. We begin by introducing what we call 'the condensed matter point of view' and reviewing some of the related concepts.

### 3.1.1 Condensed matter point of view and complexity

Since the discovery of *Anderson localisation* [21], a significant part of modern condensed matter physics has shifted its focus onto the *emergent* phenomena that is seen in the larger scales. Form this point of view, the precise microscopic details of a *system* is not too important and what matters is the cooperative effect manifest in larger distances. It is perhaps appropriate to consider this in relation to a *box* in space:[1]



Inside the box, we have some *local* quantum mechanical degrees of freedom. These degrees of freedom live *inside* the box and one can write the Hamiltonian for them and their near neighbours with no specification on what happens at a distance from the box. However, an experiment makes observations on much larger scales than this box. One may then proceed to solving these equations in attempt to decide what happens at a larger distance, but this may not be feasible. Moreover, generally in condensed matter scenarios, what is inside the box may be extremely complicated with many interacting

[1]For this analogy, we were inspired by the S. Coleman memorial talk given by E. Witten in 2005 [22].

degrees of freedom making its quantitative representation[2] even more infeasible. The science which investigates the large scale behaviour of systems (or structures) that are inherently complex is called *complexity*.

The science of complexity spans a rather wide range of fields and phenomena. In fact, there is not a consensus among physicists what *exactly* they mean when they say 'complexity'. In many cases the complexity is used to refer to macro-scale complexity arisen from micro-scale simplicity. While it may seem in contrast with our introduction just now, the general idea behind complexity science is always the same, *the evolution of a system in a way not determinable by reductionist methods, but through studying the collective behaviour and the emergent universal phenomena.*

### 3.1.2 Emergent phenomena

Perhaps one of the best understood example for such emergence phenomena is the *Anderson localisation* [21, 23] where the wave functions become localised in materials (see Section §3.2.2). The *emergence* is the phenomena in which '*more is different*' [24], *i.e.* the large scale physics is rich with *novel characteristics*[3]. One such characteristic is *universality*, namely the appearance of global features independent of the microscopic parameters of the system. Universal features are perhaps most rigorously studied in statistical mechanics along with *critical points* where the systems change their large-scale behaviour (*e.g.* phase) whilst some order parameter (*e.g.* temperature) is changed smoothly (see [25] for a review). If this change in large-scale behaviour (*i.e. phase transition*) is continuous, the distance over which fluctuations of microscopic degrees of freedom are correlated (*i.e. correlation length*) becomes effectively infinite, forcing the entire system to acquire the same phase as all the competing phases vanish. The presence of universal characteristics in a system would point to a mechanism that would allow very long distances to be correlated. Both in condensed matter physics and in statistical physics, this phenomena may also occur if there is a mechanism allowing transport throughout the system (*e.g.* dissipation or dissipative transport) or some forms of *disorder*. Another emergent characteristic is the *scaling behaviour* between variables of the complex system. The scaling in general take the form of *power laws* with exponents showing universality.

### 3.1.3 Coarse Graining and Renormalisation Group (RG)

One of the central themes in emergent phenomena and complexity is *coarse graining*. In short, coarse graining refers to integrating over (or smoothing) the smallest scales, often irrelevant fluctuations, in a system of many interacting degrees of freedom. It serves as a procedure effectively projecting the microscopic physics to larger scales which are relevant for measurements. Many phenomena may be studied qualitatively to be emergent under coarse graining as well as hierarchical levels in branches of physics.[4] For the discussion on this paper, we will be mainly interested in possibly *fundamental* properties of a system in much shorter scales, becoming *statistical* through processes akin to coarse graining. Nevertheless, it should be understood that the analogy with real physical phenomena and coarse graining grows much deeper. The framework in which these ideas are applied to a wide range of problems is called *renormalisation group* (RG). In renormalisation group studies one attempts to re-parametrise a problem in a simpler way while staying true to its physical essence. These studies are also called renormalisation group *flow* since the RG equations describe some 'trajectory' formed as a result of performing a coarse graining procedure successively.

[2] In addition to this *microscale complexity*, while the system of equations in condensed matter physics are in general known (many body Schödinger equation), this is in general not true for the early Universe physics.

[3] These characteristics are realised by cooperative behaviour and renormalisation group methods which will be discussed next.

[4] An example would be thermodynamics, being quantum field theory coarse grained, also see [26, 27] for a general discussion on these topics.

### 3.1.4 Establish links with Cosmology

Complexity science thrives on establishing analogies. Previously, we have introduced the 'condensed matter point of view' and gave definitions for various related subjects. In the preceding sections, we will make an attempt to discuss the analogies between cosmology and condensed matter physics. Let us first revisit the initial discussion about the *box* in Section §3.1.1. In order to make sense out of such analogy, one must be able to define a box, namely the theory must have some local gauge invariant parameters defined *inside* the box. In condensed matter physics this is easy since the real space parameters (*i.e.* locations, angles etc.) are in general well defined. However, in cosmology, accounting for the effect of *gravitational coupling* may complicate things. In the presence of gravity, the box itself may not be well defined since the parameters such as location in real space are not *gauge invariant* and the size of the box (*i.e.* the distance measure) changes due to metric fluctuations. We will attempt to avoid this issue in two ways. First, we will consider classical, super-horizon effects, as discussed in Section §3.4.1. Next, in our discussion of the quantum phenomena, we will employ the effective field theory formalism introduced in Section §2.5 which allows *decoupling* from gravitational effects. Hence in this way, defining some form of a box in order to calculate coarse-grained statistical effects on the cosmological observables should be possible in fundamental early Universe cosmology, as in condensed matter physics. Next, we start with a short conceptual review of *disorder* in theories of condensed matter and statistical physics.

## 3.2 Disoder or 'Ill-Condensed Matter'[5]

Disorder, or inhomogeneities, exist naturally and ubiquitously in real would quantum or classical many-body systems. Their consideration in condensed matter physics has been remarkably rewarding (see for review e.g [28–30]). In fact, many phenomena such as percolation [31], localisation [21], spin glasses [32] and topological defects [33] are discovered by considering disorder. Moreover, disorder adds much more to the phenomena of criticality that we have mentioned along with continuous phase translations (see for review *e.g.* [34, 35]). We can easily formalise disorder with a fermion $\psi$ (*e.g.* an electron) scattering by a random potential $m(\mathbf{x})$

$$\mathcal{S} = \mathcal{S}_0 - \int \mathrm{d}x \; m(\mathbf{x})\bar{\psi}\psi \;, \tag{3.1}$$

with four-vector $x := (t, \mathbf{x})$ and $\mathrm{d}x = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}$ where the $\mathcal{S}_0$ is free fermion action. Thouless in [28, 36] showed that for mass dimensions with $D \leq 2$ (*i.e.* of random potential $m(\mathbf{x})$), disorder was relevant. Next, we begin studying the mechanism of disorder.

### 3.2.1 Quenched Disorder

For our purposes, disorder can be separated into two types, *annealed* and *quenched* disorder. Annealed disorder is one where the degrees of freedom parametrising the impurities are in thermal equilibrium with the rest of the system. Or in other words, the frequency of the fluctuations *sourcing* temporary deviations from homogeneity are compatible with the system (if not faster). In such cases, system is called to be *ergodic*[6] where one can simply treat disorder and other system variables in the same footing, *i.e.* taking average over this effect gives the correct statistical measures of partition function and other quantities. There many studies of annealed statistical effects in the early Universe, see *e.g.* for inflation [37–40]. Although the technical treatment of annealed disorder is much easier, the phenomena it facilitates is limited. On the other hand, quenched disorder is a remarkably rich and curious phenomena. In systems with quenched disorder, the inhomogeneities are *frozen*, *i.e.* the degrees of freedom sourcing random fluctuations are *not* in equilibrium with the system. These systems are in

[5] This chapter title was inspired by the 1979 XXXI Session of the Les Houches Summer School [28] on 'disordered' condensed matter systems which this section partly follows.

[6] Ergodic means that the system once not disturbed and let to evolve, quickly reaches to thermal equilibrium as there is no mechanism to prohibit its thermalisation, see also Section §3.3.1.

general non-ergodic. We can imagine this effect as some field propagating in an environment with relatively constant impurities. Depending on the location and size of these impurities, the field may get distorted locally (*e.g. pinning*) for times much longer than field's fluctuation frequency. In this scenario impurities act like *potential barriers* restraining system from reaching equilibrium. Hence, one has to treat the variables associated with disorder *separately* from the systems' variables. Statistically, this complicates things considerably since the common measures such as the averaged partition function are no longer reliable as the variance of the field fluctuations are much larger. We then need to calculate other quantities commonly called *self averaging*[7] quantities such as the *free energy*. In Section §3.5 we will introduce a tool to account for this.

### 3.2.2 Percolation and localisation

For a given continuous system, percolation [41] may become synonymous with transport problems such as propagation and dissipation, as well as electrical problems such as conductivity or insulation.. The theory of critical phenomena also applies to percolation problems in which systems exhibit universal characteristics. We will see in the following sections that inflation may be studied as a percolation problem where the inflaton field driving inflation percolates through some environment. This will become much more apparent when discussing disorder in EFT of inflation Section §3.4.2. In addition to percolation, perhaps the most famous example of phenomena emergent from quenched disorder is the Anderson localisation [21]. Anderson showed in his 58' paper that if there exist sufficient amount of disorder, electron wavefunction becomes exponentially localised, *i.e.* the envelope of the wave function decays exponentially from some point in space

$$|\Psi(\mathbf{x})| \sim \exp\left(|\mathbf{x} - \mathbf{x}_0|/\xi\right) \;, \tag{3.2}$$

where $\xi$ is the correlation length (see for review *e.g.* [30]) meaning that electron eigenstate is localised at some suitable region and the system becomes an insulator. Here, this phenomena is due to quantum interference of waves which are scattered by impurities. The Anderson model for non-interacting system of spins can be written as

$$H = \sum_i \varepsilon_i \hat{a}_i^\dagger \hat{a}_i + V \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j \;, \tag{3.3}$$

where $\hat{a}$ ($\hat{a}^\dagger$) are the annihilation (creation) operators for electrons, $\langle \cdot \rangle$ refers to nearest neighbours and $\varepsilon_i$ are randomly distributed energies with some width $W$. The ratio $W/V$ measures the size of the disorder. Anderson discussed that when $W/V$ is large enough, system would become an insulator. Soon after, this phenomena has been shown to be correct especially in one-dimension, even for very weak disorder, *i.e.* all $W > 0$ by Mott [42] and later for dimensions $D \leq 2$ [23]. The localisation phenomena plays central role in a wide range of modern physics research.

## 3.3 Topics in modern condensed matter

In the previous section we introduced the concept of disorder as it was in mid-late 20th century. We will now make an attempt to discuss some of the more current research in disordered condensed matter systems and point their relevance to cosmological scenarios when possible. Especially today, the line that draws the limits of what is considered 'condensed matter physics' is of coarse as vague as it is in defining complexity. Throughout this section, we will mainly focus on subjects that can perhaps be collected under 'far from equilibrium (quantum) dynamics'. This field is perhaps one most in synch with cosmology as it is also strongly driven with the motivation of understanding various early Universe phenomena such as reheating and preheating, see *e.g.* [2, 26, 43]. In this section we qualitatively review various phenomena seen in isolated quantum many-body systems far from equilibrium.

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[7]For our purposes in this work, we define the 'self averaging quantity' as one that has vanishing variance if averaged our a sufficiently large ensemble.

[7]Free energy is the logarithm of the partition function, namely $\mathcal{F} \sim \ln \mathrm{Tr}[\exp(-\mathcal{H})]$.

### 3.3.1 Thermalisation

One of the big questions common both in cosmology and also condensed matter physics is the equilibration (or reaching the equilibrium state) of *isolated* many-body quantum systems. In a thermodynamic system, this process is called thermalisation. The thermalisation of all initial states means that all many-body eigenstates of the system's Hamiltonian $H$ are thermal. This is called Eigenstate Thermalisation Hypothesis (ETH) and has many applications in condensed matter physics, *e.g.* [44]. The subject of thermalisation of a *non-equilibrium system* was first studied in length in a cosmological context by R. Brandenberger et.al. in [2].

If an isolated system exhibits non-ergodic character at some initial time, and then reaches thermal equilibrium at some later time, this requires an *effective memory loss* of initial conditions. The transition between ergodic and non-ergodic behaviour for isolated many-body quantum systems is a very active field of research in condensed matter physics (see *e.g.* [45]) and relates directly to thermalisation process of the early Universe.

### 3.3.2 Prethermalisation

The *effective memory loss* discussed in the previous section is realised by unitary time evolution of a system with only *approximately* conserved quantities exhibiting *quasi*-stationarity while *not* in equilibrium. This phenomena is called pre-thermalisation and has many applications in cosmology, *e.g.* [46], as well as modern condensed matter systems [47]. Disorder is closely linked to thermalisation and prethermalisation. This is mainly because it may drive a system towards a non-ergodic state, resulting in the acquirement of extensive conservation laws and symmetries preventing the system from reaching a thermal state. In what follows we will try to give weight to phenomenological aspects of disordered systems in modern condensed matter physics.

### 3.3.3 Localisation revisited

Systems that do not thermalise are often called Anderson-localised systems. The phenomena is in general due to quenched disorder, see *e.g.* [45, 48, 49], (see however [50]). We already described the phenomena of single-particle localisation as suggested by Anderson [21]. The contemporary studies of single particle localisation include quantum Hall effect, disordered wires, transfer problems and many others, see for review [51]. The localisation phenomena collectively experienced by many degrees of freedom of a given system is called many-body localisation (see *e.g.* [52]). The main difference is that the many-body localisation involves *interactions* between degrees of freedom of the system (such as electron-electron interactions) and has a wide range of applications, *e.g.* see for review [53]. In what follows, we will qualitatively review an aspect of localisation and related phenomena, so-called 'emergence of slow dynamics'.

### 3.3.4 Emergence of slow dynamics

The recent studies on systems with strongly correlated degrees of freedom as well as various quenched scenarios suggest systems can be *trapped* in prethermalisation state for long times, see *e.g.* [46, 47, 54, 55]. This phenomena is sometimes called slow dynamics. If the microscopic energy scale of a system span a wide range, slow dynamics may emerge from the slow modes behaving like impurities for faster modes. This may cause the faster modes to become localised. This phenomena could also be studied from a cosmological perspective. For instance, during inflation, the long modes crossing horizon becomes frozen. The longer modes with $k \gg H$ may be considered to have similar effects on faster modes around the horizon. In the next chapter, we will study disorder classically on super-horizon scales with similar motivation. For quantum effects in considering the phenomena of slow dynamics, the presence of quenched disorder would have a direct effect on the (p)reheating phenomena suggested first in [56]. Other examples emergent slow dynamics include (nearly) integrable one-dimensional systems such as polymers, spin-chains, see e.g. [57] and disordered wires [58]. Other phenomena include Griffiths phase, which is related to slower evolution of a system around 'dirty'[8] critical points near phase transition, see *e.g.* [59].[9] The phenomena of slow dynamics is also strongly linked with cosmology in the context of the abundance of light particles and (many) broken symmetries we observe in the Universe. We will come back to this discussion later, see however [61].

### 3.3.5 Non-equilibrium universality

For equilibrium theories, universality phenomena have been in general well understood. This is *not* the case for non-equilibrium systems and the study of non-equilibrium universality classes is a very active area in condensed matter physics today. From a cosmological point of view, looking into condensed matter physics, the ultimate goal would be a low energy small-scale laboratory experiment to contribute to the understanding of the early Universe physics. Non-equilibrium universality studies have this potential (see for discussion on this topic *e.g.* [62]).

## 3.4 Analogies with inflation

It is perhaps possible to find similar analogies between condensed matter scenarios and cosmology when considering reheating (and preheating) phenomena. Nevertheless, our main interest in this work is the inflation. We begin by considering disorder as a classical effect on super-horizon scales.

### 3.4.1 Emergent slow dynamics on super-horizon scales

In Section §3.3.4 we discussed how systems with wide separation of energy scales may show emergent slow dynamics. In such systems, the slow modes act as quenched impurities on the fast modes. During inflation, this phenomena may be analogous to long-wavelength modes freezing beyond horizon and serve as a (classical) disordered landscape for the 'faster' short-wavelength modes around the horizon crossing. One must note however that many emergent phenomena such as localisation discussed throughout this section, is mostly a microscopic phenomena. This is why such effects would not be manifest in treating disorder classically. Nevertheless, even in a classical setting, quenched impurities require much different treatment than regular stochastic effects (see Section §3.5.1) and in turn may cause suppression of faster modes, hence the 'slowing' of dynamics. Perhaps this phenomena, if exist, would be partially responsible from a more 'graceful exit' at the end of inflation. We will be motivated with considering these reasons in applying our formalism for disorder on super-horizon scales in Section §4.2.

### 3.4.2 EFT of inflation

Now we turn our attention to quantum phenomenology of disorder. In this section, will add to the analogies between inflation and condensed matter phenomena by discussing two properties of EFT of inflation. EFT of inflation plays a central role in our study of disorder in the early Universe.

#### Spontaneous symmetry breaking[10]

An important analogy between condensed matter scenarios and inflation may be via considering inflation as a spontaneous symmetry breaking in time. It is quite intuitive in cosmology to assume there exist a preferred direction, *i.e.* forward in time. Nevertheless, realising inflation as a mechanism that spontaneously breaks the time diffeomorphisms is perhaps more involved. In order to see this, recall the FRW background with nearly constant expansion rate $|\dot{H}| \ll H^2$, which suggests an approximate time-translation invariance of the background. However, from a purely phenomenological standpoint,

[8] Here, 'dirty' critical point is a critical point of the system near some phase transition under the effect of disorder.

[9] Here, we wish to return to our previous analogy with cosmology, *i.e.* the correspondence of inflaton propagation with percolation problem in one-dimension. One-dimensional condensed matter systems such as wires is very suggestive for making further analogies with cosmology. A detailed analysis on this subject was recently done in [60].

[10] This section was mainly inspired by D. Baumann's notes on inflation [63].

we *know* that the inflation has to come to an end, hence this symmetry has to be spontaneously broken. This spontaneous breaking of time-translation symmetry introduces the Goldstone boson[11]

$$U := t + \pi(x) \ , \tag{3.6}$$

where we performed a spacetime-dependent time shift. Under time transformation $t \to t+\xi$, Goldstone field transforms as $\pi \to \pi - \xi$ keeping $U = t + \pi$ invariant. The Lagrangian for the Goldstone boson then becomes

$$\mathcal{L} = F(U, (\partial_\mu U)^2, \Box U, \dots) \ . \tag{3.7}$$

At first order in energy, the Lagrangian can be written as

$$\mathcal{L} = \Lambda^4(U) - f^4(U) g^{\mu\nu} \partial_\mu U \partial_\nu U \ , \tag{3.8}$$

where $\Lambda(U)$ and $f(U)$ are free functions. In order to cancel the linear terms (*tadpoles*) from the action, we fix these coefficients by the de Sitter background $H(t)$,

$$\Lambda^4 := -M_{pl}^2(3H^2 + \dot{H}) \quad \text{and} \quad f^4 := M_{pl}^2 \dot{H} \ , \tag{3.9}$$

and get the Lagrangian

$$\mathcal{L} = M_{pl}^2 \dot{H} g^{\mu\nu} \partial_\mu U \partial_\nu U - M_{pl}^2(3H^2 + \dot{H}) \ . \tag{3.10}$$

This Lagrangian is equivalent to the slow-roll inflation $\mathcal{L}_\phi$ introduced in Section §2.3.4 with $\phi = \dot{\bar{\phi}}(t+\pi)$ and $V(\phi) = M_{pl}^2(3H^2 + \dot{H})$. In fact in this formalism, the Goldstone boson $\pi$ becomes an *order parameter* as it parametrises the field of fluctuations in the 'clock' measuring *time* during inflation $\pi \sim \delta t$. This is analogous to spontaneous symmetry breaking in condensed matter systems, where the spatial symmetry is spontaneously broken in the direction of growth. In condensed matter systems, symmetry breaking is induced by some external mechanism (*e.g.* applied voltage). In the context of inflation, symmetry is spontaneously broken due to the *fact* that inflation comes to an end. While in inflation we have spontaneously broken time diffemorphisms, in condensed matter physics, it is the *spatial* diffeomorphisms that are spontaneously broken. This *duality* in time and space between EFT of inflation and condensed matter physics is an exciting observation.

### Decoupling from gravity

We began this chapter by discussing the success of any quantum scale analogy between cosmology and condensed matter physics would depend on how well one can describe a local degree of freedom.[12] This is a complicated issue and is central to most contemporary research in theoretical physics. Here, we consider EFT of inflation as a way to avoid this problem. We begin with the observation that due to *Noether's theorem*, Goldstone boson has two distinctive properties. First, the energy of the Goldstone

[11] The *Noether's theorem* tells us that 'for every symmetry global *continuous* symmetry of the action, there exist a conserved current $j^\mu$, with

$$\partial_\mu j^\mu = 0 \ . \tag{3.4}$$

The *Goldstone's theorem* states that the spontaneous breaking of a global continuous symmetry leads to a massless Goldstone boson $\pi$. Goldstone modes $|\pi\rangle$ are obtained by performing symmetry transformations on the ground state with a spacetime dependent transformation parameter

$$U = e^{i\pi(x)/f_\pi}, \qquad \text{where} \qquad \pi := \pi^a T^a \ . \tag{3.5}$$

[12] Throughout this work we avoided theories of information entropy and entanglement complexity that often make attempts to better describe or answer this issue. In fact most of the recent research has completely abandoned the use of locality or even spacetime. Since condensed matter physics is basically a boundless field, these theories are also well within the range of condensed matter physics. Here, we we remain in the more phenomenological side of the picture and not discuss these theories.

must go to zero in the limit

$$\lim_{p\to 0} E(p) = 0 \ , \tag{3.11}$$

where $p$ is the 3-momentum. This can be used to show that Goldstone bosons are *massless*. The next property is that the massless-ness of the Goldstone boson can be extended into all interactions in the limit $p \to 0$ (see for a review [12, 64]). For the effective field theory of inflation (see Section §2.5) this constitutes to the *decoupling limit* where $M_{pl} \to 0$ and $\dot{H} \to 0$ while holding $M_{pl}^2|\dot{H}| \gg H^4$ fixed. In doing so, we completely decouple the Goldstone boson from gravity! This property indeed makes the EFT of inflation formalism very attractive for considering condensed matter phenomena. In the next section we will review the applications of stochastic potentials in inflation.

### 3.4.3 Random potential

Lastly in this section, we will review the coarse-grained stochastic contribution of some fundamental microscopic physics in the *subhorizon* dynamics of the early Universe. This is realised as a *stochastic* potential coupled to the fields during inflation (see *e.g.* [39, 40, 65]). At the fundamental level, this contribution may be unavoidable considering a string theory landscape with very large number of moduli (see *e.g.* [66–68]), or a theory with multiple fields and complicated couplings, self-interactions, etc., contributing to the inflation mechanism. In this aspect, these theories are analogous to condensed matter physics where the parametrisation of the stochasticity is the ubiquitous first step in most quantitative consideration. Moreover, this analogy becomes more apparent once we consider inflaton propagation as a particle propagation in a condensed matter system. We demonstrate this in Figure 3.1. From an EFT of inflation perspective Goldstone boson can easily be thought as similar to an electron in a one dimensional wire. The stochastic contribution can then be thought as impurities on the path of inflaton. In what follows, we will give one example of an inflationary model with similar formalism, the *trapped inflation* [69].

#### Trapped inflation

In trapped inflation [69], one has a mechanism in which inflaton dumps up its kinetic energy into production of other particles while rolling down a steep potential. Similar phenomena was originally studied under (p)reheating scenarios [2, 61, 70, 71] which has been central to the contemporary research in condensed matter physics as well as early Universe cosmology (see §3.3.2). This mechanism can be realised by considering a coupling term in the Lagrangian

$$\frac{1}{2} g^2 \sum_i (\phi - \phi_i)^2 \chi_i \ , \tag{3.12}$$

where $\phi_i$ represents the *points* on the inflaton's path where the corresponding particles $\chi_i$, become very light and are produced. The Lagrangian describing the this mechanism is written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \frac{1}{2}\sum_i \left(\partial_\mu\chi_i\partial^\mu\chi_i - g^2(\phi - \phi_i)^2\chi_i^2\right) + \dots \ , \tag{3.13}$$

and the expectation value of the number density of produced particles $\chi_i$ is given as [69]

$$n_\chi(t) \simeq \frac{g^{3/2}}{(2\pi)^3} (\dot{\phi}_i)^{3/2} \frac{a(t_i)^3}{a(t)^3} \ , \tag{3.14}$$

where $\phi(t_i) = \phi_i$ and the dilution of particles due to inflation is accounted for with the scale factor $a$ (also see [61]). This process can be thought to be analogous to localisation phenomena due to impurities. Quite similarly, particle production would slow the inflaton $\phi$ as it propagates through

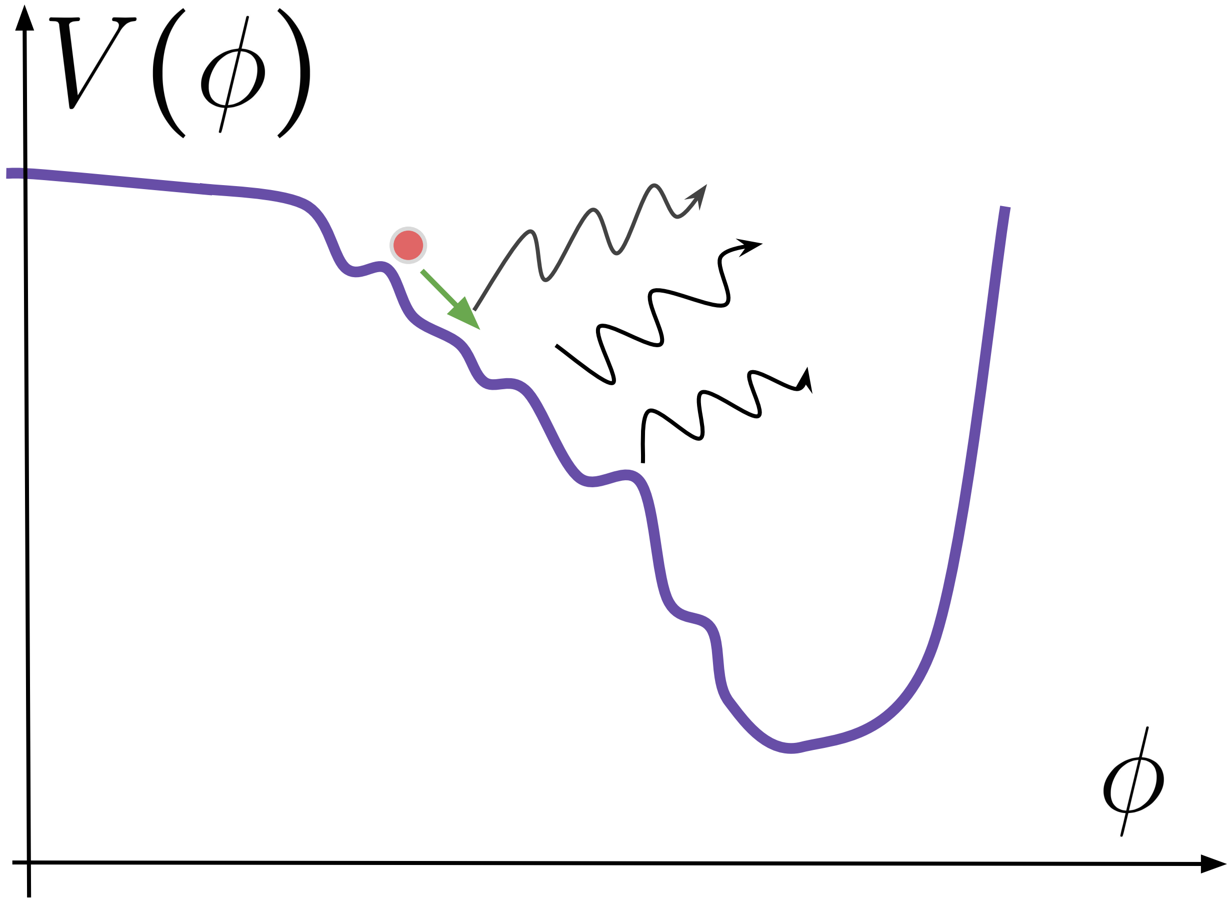


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Figure 3.1: In this figure we represent the inflaton as it propagates downwards in a disordered medium. In many cases, the impurities on the path of inflaton is considered to have a statistical effect. This is realised by introducing complicated potential terms and/or coupling inflaton field to other fields. In the latter case this may result in particle production. These particles would in turn be diluted due to exponential expansion of space. This effect may also serve to slow inflaton as it propagates through the slope.

the impurities.[13] The duality between particle production in the early Universe and localisation was also discussed in a recent paper by D. Baumann et. al [60]. There the authors found for a multi-field scenario, the *variance* of the total occupation number of fields after $N$ particle production is *only weakly* dependent on the number of fields. We will return to these observations later in Section §4.3.1. Next, we will discuss one of the formalisms available to studying quenched disorder.

## 3.5 Tools for calculating effects of *quenched* disorder

In this section we will introduce one of the tools necessary to calculate propagagators for a system with disorder, which is simply a trick for now, *i.e.* the *replica trick*.[14] We will then move on to describe a non-perturbative variational approach to calculate the system's propagator. Our review in this section will be limited to classical methods.

[13]Note that we are discussing quantum effects of considering disorder in these sections. The classical analogue was discussed in Section §3.4.1.

[14]Another available method known to the author is the supersymmetry method in [72, 73] which extends to random matrix theories (RMT) [74]. Here we will focus on the replica method instead.

### 3.5.1 Replica trick

The replica trick was perhaps first given considerable attention in condensed matter physics with the first developments in *spin glass* theories [75]. It is based on the simple identity

$$\boxed{\ln \mathcal{Z} = \lim_{N\to 0} \frac{1}{N}\left(\mathcal{Z}^N - 1\right)\ .} \tag{3.15}$$

The relevance and necessity of defining this identity is as follows: In systems with quenched disorder, the degrees of freedom associated to the impurities are static in the time frame of systems' degrees of freedom (see Section §3.2.1). This means that we cannot treat the two in the same footing, *i.e.* the result of taking an average of any given measure over disorder will highly vary[15] and it is necessary to average the free energy, left hand side of Eqn. (3.15), instead of the partition function itself. However it turns out that this requires performing a logarithm on each realisation of the disorder. What replica trick does is to create $N$ identical copies, or *replicas*, of the system. In doing so, we get rid of the logarithm and end up with a partition function for $N-$replica fields, $\mathcal{Z}^N$. Note however, at the heart of this simple mathematical manipulation, there is the assumption that $N$ is an integer and we are allowed to analytically continue $N \to 0$ which has remained a widely discussed issue since seventies, see *e.g.* [76, 77]. Here, we will take the point of view that, accounting for the success of particularly spin glass theories, it is our incapacity failing to resolve this issue rather than the invalidity of the approach [78]. The replica method is at the centre of spin glass theories and has been a significant part of many condensed matter theories involving quenched disorder[16].

### 3.5.2 Replica approach

In what follows, we will introduce the so called replica field theory first introduced in [79].

#### Scalar Model[17]

Let us consider some Hamiltonian for a scalar field $\varphi(x)$ with a kinetic, mass and potential terms

$$\mathcal{H}_\varphi = \frac{1}{2}\int \mathrm{d}^D x \left[\sum_\mu^D (\partial_\mu \varphi)^2 + m\varphi^2\right] + \int \mathrm{d}^D x\ V(\varphi;x)\ , \tag{3.16}$$

where we have taken a general $D$ dimensional manifold for the time being. For the rest of the paper, we will use Einstein notation, *i.e.* repeated indices will be summed up. The potential $V(\varphi;x)$ is a *quenched* random variable which *can* depend on the scalar field $\varphi$ and also some coordinates $x_i$. Moreover, we assume the random potential is governed by a Gaussian distribution of zero mean. It is common in condensed matter physics to get difference correlations for the random potential in the form

$$\begin{aligned} \overline{V(\varphi;x)V(\varphi';x')} &= \int \mathcal{D}[V]P[V]\ V(\varphi;x)V(\varphi',x') \\ &= \delta^D(x-x')\ \mathcal{F}\left(|\varphi-\varphi'|^2\right)\ , \end{aligned} \tag{3.17}$$

where $\varphi' := \varphi(x')$ and $\overline{(\ \cdot\ )}$ refers to the average over disorder. We also take the integration measure to obey normalisation condition $\int \mathcal{D}[V]P[V] = 1$ .

[15]Perhaps a good way to understand this is from an experimental point of view: In the presence of relatively static impurities in a system, making a measurement of a quantity that is fluctuating with the system's variables will itself give largely fluctuating results. One solves this 'measurement' problem in statistical physics with *self-averaging* quantities such as the *free energy* which is the *logarithm* of the partition function.

[16]*e.g.* See (non-linear) sigma-models for disordered electronic systems [29].

[17]In [79], method is applied on a $D$ dimensional manifold represented by a $N$ component vector field in a $N+D=d$ dimensional space. Here, we shall consider a scalar field instead.

### Replicated Hamiltonian

Let us now apply the replica method as in Eqn. (3.15). The replicated partition function, $\mathcal{Z}^N$ takes the form

$$\overline{\mathcal{Z}^N} = \int \mathcal{D}[V]P[V] \int \mathcal{D}[\{\varphi\}] \exp\Big( - \mathcal{H}_N[\{\varphi\}]\Big) \ , \tag{3.18}$$

where $\int \mathcal{D}[\varphi] = \int \prod_i^N [\varphi_i]$ and Hamiltonian in replica space $\mathcal{H}_N[\{\varphi\}] = \sum_{a=1}^N \mathcal{H}[\varphi_a]$ takes the form

$$\mathcal{H}_N[\{\varphi\}] = \frac{1}{2} \int \mathrm{d}^D x \ \sum_{a=1}^N \left[\partial_\mu \varphi_a \partial^\mu \varphi_a + m\varphi_a^2\right] + \int \mathrm{d}^D x \ \sum_{a=1}^N V(\varphi_a; x) \ . \tag{3.19}$$

From a field theoretical point of view, replicating the Hamiltonian amounts to generating $N-$copies of the theory where $\varphi_{i=\{1,\dots,N\}}$ now representing a field in the Hilbert space of $i$th theory. The *full* theory represented by the collection of all fields in the limit $N \to 0$. The purpose of this manipulations is to get a measure in which the effects of disorder is manifest.

Previously we have discussed the necessity of considering free energy and hence introduced the replica method. However since our Gaussian distributed random potential in Eqn. (3.19) has zero mean,[18] averaging over it will not give any information about the properties of the disordered system. In order to solve this issue, we will now use the Gaussian characteristic of the potential term. Let us omit writing the field dependence of the potential term for simplicity and define a correlation function $R(x - x')$ in the form

$$\overline{V(x)} = 0 \qquad \overline{V(x)V(x')} = R(x - x') \ . \tag{3.20}$$

The Gaussian distribution of the random potential can be written as follows

$$P[V] = \exp\left(\frac{1}{2} \int \mathrm{d}^D x \int \mathrm{d}^D x' \ R^{-1}(x - x')V(x)V(x')\right) \ , \tag{3.21}$$

where the integral kernel $R^{-1}(x - x')$ is the functional inverse of the correlation function,

$$\int \mathrm{d}^D x'' R^{-1}(x - x'')R(x'' - x') = \delta^D(x - x') \ . \tag{3.22}$$

We now wish to calculate the moments of the distribution $P[V]$. This is done by first computing the generating functional

$$\mathcal{I}[J] = \int \mathcal{D}[V]P[V] \exp\left\{\int \mathrm{d}^D x \ J(x)V(x)\right\} \ , \tag{3.23}$$

where $J(x)$ is some external current, and observing that this expression is equivalent to

$$\mathcal{I}[J] = \exp\left\{-\frac{1}{2} \int \mathrm{d}^D x \int \mathrm{d}^D x' J(x)J(x')R(x - x')\right\} \ , \tag{3.24}$$

once the random potential is defined with Gaussian distribution as in Eqn. (3.21). Using this result and also reintroducing the field dependence back to the correlation function in the form Eqn. (3.17), we can write the random term in our replicated Hamiltonian simply as,

$$\boxed{\mathcal{H}_N[\{\varphi\}] = \frac{1}{2} \int \mathrm{d}^D x \ \sum_{a=1}^N \left[\partial_\mu \varphi_a \partial^\mu \varphi_a + m\varphi_a^2\right] - \frac{1}{2} \int \mathrm{d}^D x \ \sum_{a,b=1}^N \mathcal{F}\left(|\varphi_a - \varphi_b|^2\right) \ .} \tag{3.25}$$

The perturbative treatment of the given disorder in condensed matter physics is to expand $\mathcal{F}$ in powers

[18] Even in the case of a potential with non-zero mean value, this can always be shifted to zero.

of $\varphi$. Up to quadratic part, this results in a free propagator $F_{ab}$ where [79]

$$F_{ab}(k) = \frac{\delta_{ab}}{k^2 + N2\mathcal{F}'(0)} + \frac{2\mathcal{F}'(0)}{k^2(k^2 + N2\mathcal{F}'(0))} , \tag{3.26}$$

which for $N \to 0$ introduces a correction term $1/k^4$. Next, we will introduce a variational method in order to go beyond the perturbative results.

### 3.5.3 Variational method

**Variational Hamiltonian**

In order to do non-perturbative calculations for such disordered condensed matter systems one uses the *variational method* first introduced by Feynman in 55' [80]. In what follows, lets consider classical statistics and confine ourselves to a 3-dimensional manifold for simplicity. The first step is to define a variational Hamiltonian for the theory with multiple replicas, *i.e.* for the action in Eqn. (3.25)

$$\mathcal{H}_{\text{var}} = \frac{1}{2}\int \mathrm{d}^3x \sum_{a=1}^{N} \left[\partial_\mu\varphi_a\partial^\mu\varphi_a + m\varphi_a^2\right] - \frac{1}{2}\int \mathrm{d}^3x \sum_{a,b=1}^{N} \sigma_{ab}\ \varphi_a\ \varphi_b , \tag{3.27}$$

where the variational Hamiltonian depends on the $(N\times N)$ symmetric $\sigma_{ab}$ matrix also called the *replica structure*. It is common the write the variational Hamiltonian by introducing the propagator

$$G_{ab}(k) = \left[(k^2 + m)\mathtt{I} - \sigma\right]^{-1}_{ab} \tag{3.28}$$

where $\mathtt{I}$ is the identity matrix and we have taken the Fourier transformation of the free part of the action, *i.e.* $k^2 = |\mathbf{k}|^2$. Variational Hamiltonian becomes

$$\mathcal{H}_{\text{var}} = \frac{1}{2}\sum_{a,b=1}^{N}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\ \varphi_a(k)\ G^{-1}_{ab}(k)\ \varphi_b(-k) , \tag{3.29}$$

where we have conserved the momentum.

**Saddle point approximation**

Next step in the variational method is to calculate the *free energy* of the theory. Free energy has a central importance in many statistical and condensed matter physics theories due to it being a *self-averaging* quantity. For our purposes later in the next chapter, we shall not concern ourselves much with this reasoning, however we will explain its importance in short here from the condensed matter physics perspective. First, let us define the inequality [80]

$$\boxed{F \geq \langle\mathcal{H}_{\text{var}} - \mathcal{H}_n\rangle_{\text{var}} + F_{\text{var}}\ ,} \tag{3.30}$$

where $F$ is the free energy of the *full* theory and $F_{\text{var}} = \ln\mathcal{Z}_{\text{var}}$ . We shall first calculate the r.h.s of this equation. Since we will then follow with a saddle point approximation $\partial F/\partial G_{ab} = 0$ we will not worry with the constant terms. First term we shall calculate is the variation expectation value of non-diagonal term in the replicated Hamiltonian $\langle\mathcal{H}_N\rangle_{\text{var}}$

$$\left\langle \int \mathrm{d}^3x \sum_{a,b=1}^{N} \mathcal{F}\Big(\Big|\varphi_a - \varphi_b\Big|^2\Big)\right\rangle_{\text{var}} . \tag{3.31}$$

This is done by expanding the correlation functional $\mathcal{F}$ in its powers and calculating the expectation value for each term. Summarising in few steps

$$\begin{aligned}\int \mathrm{d}^3x \sum_{a,b=1}^{N} \mathcal{F}\left(|\varphi_a - \varphi_b|^2\right) &= \int \mathrm{d}^3x \sum_{a,b=1}^{N}\sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}}{n!} |\varphi_a - \varphi_b|^{2n} \\ &= \sum_{a,b=1}^{N}\sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}}{n!}(2n-1)!! \left| \int \frac{\mathrm{d}^3k}{(2\pi)^3}\varphi_a(k)\varphi_a(-k) + \varphi_b(k)\varphi_b(-k) - 2\varphi_a(k)\varphi_b(-k)\right|^n ,\end{aligned} \tag{3.32}$$

and taking the expectation w.r.t variational Hamiltonian introduced in Eqn. (3.29)

$$\begin{aligned}\langle \text{ Eqn. (3.32) } \rangle_{\text{var}} &= \sum_{a,b=1}^{N}\sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}}{n!}(2n-1)!! \left| \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left(G_{aa}(k) + G_{bb}(k) - 2G_{ab}(k)\right)\right|^n \\ &= \tilde{\mathcal{F}}\left( \left| \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left(G_{aa}(k) + G_{bb}(k) - 2G_{ab}(k)\right)\right| \right) .\end{aligned} \tag{3.33}$$

The expectation value of the free part of the replicated Hamiltonian

$$\frac{1}{2}\int \mathrm{d}^3x \sum_{a=1}^{N} \left\langle \left[\partial_\mu\varphi_a\partial^\mu\varphi_a + m\varphi_a^2\right]\right\rangle_{\text{var}} = \frac{1}{2}\sum_{a=1}^{N}\int \frac{\mathrm{d}^3k}{(2\pi)^3}(k^2+m)G_{aa}(k) . \tag{3.34}$$

Finally, the third term involves the partition function for the variational action. This can be written as

$$\mathcal{Z}_{\text{var}} \propto \sqrt{\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathbf{det}[G_{ab}(k)]} , \tag{3.35}$$

with which the contribution to the inequality becomes

$$\ln\{\mathcal{Z}_{\text{var}}\} = c_{\text{var}} + \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathbf{Tr}[\![\ln\{\underline{\underline{\mathbf{G}}}(k)\}]\!] . \tag{3.36}$$

Combining these contributions in Eqn. (3.30) and taking a saddle-point approximation, *i.e.* $\partial F/\partial G_{ab} = 0$ we write

$$\begin{aligned}\frac{\partial}{\partial G_{cd}}\Bigg[\Bigg(\int \frac{\mathrm{d}^3k}{(2\pi)^3}(k^2+m)\sum_{a=1}^{n} G_{aa}(k) - \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathbf{Tr}[\![\ln\{\underline{\underline{\mathbf{G}}}(k)\}]\!]\Bigg) \\ + \sum_{a,b=1}^{n} \tilde{\mathcal{F}}\left(\left|\int \frac{\mathrm{d}^3k}{(2\pi)^3}(G_{aa}(k) + G_{bb}(k) - 2G_{ab}(k))\right|\right)\Bigg] = 0.\end{aligned} \tag{3.37}$$

If we take the derivation in the form

$$\frac{\partial G_{ab}(k)}{\partial G_{cd}(k')} = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{ab}\delta_{bd} , \tag{3.38}$$

we arrive at the final result

$$\begin{aligned}\sigma_{ab} &= 2\tilde{\mathcal{F}}'\left(\left|\int \frac{\mathrm{d}^3k}{(2\pi)^3}(G_{aa}(k) + G_{bb}(k) - 2G_{ab}(k))\right|\right) \quad a \neq b \\ \sigma_{aa} &= -\sum_{b(\neq a)}^{N} \sigma_{ab} ,\end{aligned} \tag{3.39}$$

where $\tilde{\mathcal{F}}'(x) = \partial\tilde{\mathcal{F}}(x)/\partial x$ . Above equations, along with the definition of the propagator in Eqn. (3.28) are the saddle point equations. With this definition of the replica structure, $\underline{\underline{\mathbf{G}}}(\mathrm{k})$ becomes the variational ansatz for the full Green's function of the system. In this representation, $\sigma_{ab}$ mimics the *irreducible self energy*, $\Sigma(t,\mathrm{k})$, which is the sum of the contributions of all diagrams with two legs such that the diagram entry cannot be separated from the exit by cutting a line of the diagram. This can be written analytically as

$$\begin{aligned} G(t,\mathbf{k}) &= G_0(t,\mathbf{k}){+}G_0(t,\mathbf{k})\Sigma(t,\mathbf{k})G_0(t,\mathbf{k}) \\ &\quad + G_0(t,\mathbf{k})\Sigma(t,\mathbf{k})G_0(t,\mathbf{k})\Sigma(t,\mathbf{k})G_0(t,\mathbf{k}) + \ldots \\ &= G_0(t,\mathbf{k}){+}G_0(t,\mathbf{k})\Sigma(t,\mathbf{k})G(t,\mathbf{k}), \end{aligned} \tag{3.40}$$

which has the solution

$$G(t,\mathbf{k}) = (G_0^{-1}(t,\mathbf{k}) - \Sigma(t,\mathbf{k}))^{-1}. \tag{3.41}$$

In short, replica approach and variational method together correspond to the self-consistent Hartree approximation. It is perhaps appropriate now to discuss the significance of this method in condensed matter physics. What replica field theory does is in fact to parametrise the minimum free energy landscape. In our formalism, we have seen that the replicated potential term in the Hamiltonian, as in Eqn. (3.25), is *attractive*. This means that the replicated fields attract one another towards the minimum free energy state since they share the same Hamiltonian.

### Replica symmetric ansatz

Finally, lets calculate the simplest approach we can take in calculating the propagator. This is to assume full replica symmetry where the replica structure satisfies the relation

$$\sigma_{ab} = \sigma \qquad \text{for all } a \neq b \ . \tag{3.42}$$

From Eqn. (3.39) and (3.42) we define the diagonal corrections as $\sigma_{aa} = \tilde{\sigma} = -N\sigma$. Propagator becomes

$$G_{ab}^{-1}(k) = \left[(k^2+m)+\tilde{\sigma}\right]\delta_{ab} - (1-\delta_{ab})\ \sigma \ . \tag{3.43}$$

We use the equality $\sum_{b=1}^{N} G_{ab}G_{bc}^{-1} = \delta_{ac}$ and the arithmetic form for the propagator $G_{ab} = A\delta_{ab} + B$ to get

$$G_{ab}(k) = \left[(k^2+m)+N\sigma\right]^{-1}\delta_{ab} + \left[(k^2+m)(k^2+m+N\sigma)\right]^{-1}\sigma \tag{3.44}$$

In the next chapter we will derive the important object, $\mathrm{Tr}[[\mathrm{G}(k)]]$, that mimics the two-point correlation function of the system in the limit $N \to 0$. With the compact notation $G_0^{-1}(k) = (k^2+m)$ and the simple full symmetric ansatz, this becomes

$$\lim_{N\to 0}\frac{1}{N}\mathrm{Tr}[[\mathrm{G}(k)]] = G_0(k) + \sigma G_0^2(k) \ . \tag{3.45}$$

In the next Chapter, we will apply this formalism to inflation and calculate the corrections to the correlation function once disorder is accounted for.

# Chapter 4

# Applications and Formalisms

**Guideline to this chapter**

In the previous chapter we introduced the analogies between condensed matter physics and early Universe cosmology. We have discussed *quenched disorder* as the mechanism that gives ground to many interesting emergent phenomena. In order to study quenched disorder in cosmology, however, we have to first extend the available formalism. The following three sections correspond to first few steps towards this direction. In the first section we introduce, for a system with quenched disorder, a general expressions for the two point correlation function, which plays a central role in cosmology. In the second section we apply a classical formalism from condensed matter physics to cosmology and calculate corrections from quenched disorder to the correlation functions on super-horizon scales during inflation. In the third section we take a look into the quantum field theory analogue of studying scenarios with disorder. There, we also point out some of the common elements between studying path integrals in cosmology and condensed matter physics. Throughout this chapter we will focus mainly on introducing formalisms. We leave more focus on cosmology for disorder scenarios for future work.

## 4.1 The two-point correlation function with replicas

We begin by calculating correlation functions for a theory with replicated fields. We give a general formalism which can both be applied to classical and quantum theories.

### 4.1.1 Connected $n$-point correlation function

The classical statistical physics (*i.e.* thermodynamics) is given by $D$-dimensional functional integrals in which the partition function,

$$\mathcal{Z} = \mathrm{Tr}[\exp\{-\beta H\}] \ , \tag{4.1}$$

plays a central role[1]. In many-body quantum statistical systems, this becomes a $D+1$ dimensional functional integral. In quantum field theory, the formulation is exactly same with the statistical systems where the partition function serves as the generating functional for correlation functions in the presence of some arbitrary source terms

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \exp\left\{-\mathcal{S}_\phi + \int \mathrm{d}x \ \phi(x)J(x)\right\} \ , \tag{4.2}$$

As we discussed in the previous chapter, it is often preferable to calculate the logarithm of this value instead. This is called *free energy* in thermodynamics and it is a self-averaging quantity. In the

[1] Trace simply means that we integrate over all microscopic degrees of freedom

language of quantum theories, this amounts to using the generating functional for $n$-point *connected* correlation functions,

$$\mathcal{W} = \ln \mathcal{Z} \ . \tag{4.3}$$

The $n$-point connected correlation functions are calculated via taking functional derivatives of $\mathcal{W}[J]$ with respect to some source term $J(x)$. Setting $J = 0$ gives the 1-point function

$$\left.\frac{\delta \mathcal{W}[J]}{\delta J(x)}\right|_{J=0} = \langle \phi(x) \rangle \tag{4.4}$$

where $\varphi(x) := \langle \phi(x) \rangle$ can be some average (*e.g.* background) value of the systems variable $\phi(x)$. The second derivative gives

$$\left.\frac{\delta^2 \mathcal{W}[J]}{\delta J(x) \delta J(y)}\right|_{J=0} = \langle \mathrm{T}\{\phi(x)\phi(y)\} \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle = \langle \mathrm{T}\{\phi(x)\phi(y)\} \rangle^c \ , \tag{4.5}$$

where $c$ stands for *connected* correlation function. The expression $\mathrm{T}\{\dots\}$ represents the *time-ordering* operator for quantum fields. We will omit showing this explicitly in the following expressions for simplicity, though it should be assumed to exist if the theory is considered quantum. In the next few sections we will calculate the two-point correlation function by simply promoting our formalism introduced in Section §3.5 to $3+1$ dimensions in Minkowski spacetime $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Instead of the Hamiltonian, we take the *action* and use the 4-vector $x = (t, \mathbf{x})$ .

### 4.1.2 Replica Trick

In the presence of disorder, we are interested stochastic average[2] of the generating functional

$$\overline{\mathcal{W}} = \overline{\ln\{\mathcal{Z}[J]\}} \ . \tag{4.7}$$

Let us use a variation of the replica trick introduced in the previous chapter (see also [81]),

$$\begin{aligned} &\frac{\delta^n}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} \overline{\ln\{\mathcal{Z}[J]\}} \\ &\quad = \lim_{N\to 0} \frac{N}{N} \frac{\delta^n}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} \overline{\ln\{\mathcal{Z}[J]\}} \\ &\quad = \lim_{N\to 0} \frac{1}{N} \frac{\delta^n}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} \ln\{1 + N\overline{\ln\{\mathcal{Z}[J]\}}\} \\ &\quad = \lim_{N\to 0} \frac{1}{N} \frac{\delta^n}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} \ln\{\overline{\exp\{N\mathcal{Z}[J]\}}\} \\ &\quad = \lim_{N\to 0} \frac{1}{N} \frac{\delta^n}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} \ln\{\overline{\mathcal{Z}^N[J]}\} \ . \end{aligned} \tag{4.8}$$

Next, we will calculate the two point correlation function using the above relation in the form:

$$\left.\frac{\delta^2 \overline{\ln\{\mathcal{Z}\}}}{\delta J(x_1) J(x_2)}\right|_{J=0} = \lim_{N\to 0} \frac{1}{N} \left.\frac{\delta^2 \ln\{\overline{\mathcal{Z}^N}\}}{\delta J(x_1) J(x_2)}\right|_{J=0} \ . \tag{4.9}$$

---

[2] As a reminder, the stochastic average is defined as

$$\overline{(\ \cdot\ )} = \int \mathcal{D}[V] P[V] \ (\ \cdot\ ) \ , \tag{4.6}$$

where $V$ is the stochastic potential term.

### 4.1.3 Two-point correlation function

We take the generator for the correlation function in the form

$$\overline{\mathcal{Z}^N[\{\phi\},J]} = \int \mathcal{D}[V]P[V] \int \prod_{c=1}^{N} \mathcal{D}[\phi_c] \exp\left\{-\sum_{a=1}^{N} \mathcal{S}[\phi_a] + \sum_{a=1}^{N} \int dx J(x)\phi_a(x)\right\}. \tag{4.10}$$

where $\int \mathrm{d}x = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}$. With the above definition, we can calculate the resulting *connected* correlation function from Eqn. (4.5) for the theory with multiple *replicas*,

$$\lim_{N\to 0} \frac{1}{N} \frac{\delta^2 \ln\{\overline{\mathcal{Z}^N}\}}{\delta J(x_1) J(x_2)}\Big|_{J=0} = \lim_{N\to 0} \frac{1}{N}\left[\overline{\left\langle \sum_a^N \phi_a(x) \sum_b^N \phi_b(y)\right\rangle_{\mathcal{Z}^N}} - \overline{\left\langle \sum_a^N \phi_a(x)\right\rangle_{\mathcal{Z}^N}} \cdot \overline{\left\langle \sum_b^N \phi_b(y)\right\rangle_{\mathcal{Z}^N}}\right] \tag{4.11}$$

where $\langle \dots \rangle_{\mathcal{Z}^N}$ indicates that we are to calculate a path integral with respect to replicated partition function $\mathcal{Z}^N$ as shown before. We write this explicitly for $J = 0$,

$$\begin{aligned}
&\lim_{N\to 0} \frac{1}{N}\left[\overline{\left\langle \sum_a^N \phi_a(x) \sum_b^N \phi_b(y)\right\rangle_{\mathcal{Z}^N}} - \overline{\left\langle \sum_a^N \phi_a(x)\right\rangle_{\mathcal{Z}^N}} \cdot \overline{\left\langle \sum_b^N \phi_b(y)\right\rangle_{\mathcal{Z}^N}}\right] \\
&= \lim_{N\to 0} \frac{1}{N}\left[\overline{\frac{1}{\mathcal{Z}^N} \int \prod_{c=1}^{N} \mathcal{D}[\phi_c] \left(\sum_{a=1}^{N} \phi_a(x)\right) \cdot \left(\sum_{b=1}^{N} \phi_b(x)\right) \exp\left\{-\sum_{b=1}^{N} \mathcal{S}[\phi_b]\right\}} \qquad \text{(L2)}\right. \\
&\left. - \overline{\frac{1}{\mathcal{Z}^{N^2}}} \underbrace{\overline{\int \prod_{c=1}^{N} \mathcal{D}[\phi_c] \left(\sum_{a=1}^{N} \phi_a(x)\right) \exp\left\{-\sum_{b=1}^{N} \mathcal{S}[\phi_b]\right\}}}_{\text{scales with } \propto N} \cdot \underbrace{\overline{\int \prod_{d=1}^{N} \mathcal{D}[\phi_d] \left(\sum_{b=1}^{N} \phi_b(x)\right) \exp\left\{-\sum_{b=1}^{N} \mathcal{S}[\phi_b]\right\}}}_{\text{scales with } \propto N}\right]
\end{aligned} \tag{4.12}$$

It makes sense to study the terms in the above equation (4.12) in terms of their behaviour in the limit $N \to 0$.[3] The term in the third line of the above equation scales as $\propto N^2$ and hence approaches zero in the limit $N \to 0$ in $\mathcal{O}(N)$. The term in the second line requires further attention. Observing the expression in the second line of equation (4.12), we can see that it can be separated into two parts, one with identical replica indices for both replica fields in the numerator, and one with unequal replica indices. For simplifying the notation, we will use the fact that the path integral is independent of the exact value of the replica indices, but focus on the cases where the replica indices are the same or different at the limit $N \to 0$.

$$\begin{aligned}
\text{L2} = \lim_{N\to 0} \frac{1}{N}&\left[\underbrace{\overline{\frac{1}{\mathcal{Z}^N} \int \prod_{c=1}^{N} \mathcal{D}[\phi_c] \left(\sum_{a=1}^{N} \phi_a(x)\ \phi_a(y)\right) \exp\left\{-\sum_{b=1}^{N} \mathcal{S}[\phi_b]\right\}}}_{\text{scales with } \propto\ N}\right. \\
&\left. + \underbrace{\overline{\frac{1}{\mathcal{Z}^N} \int \prod_{c=1}^{N} \mathcal{D}[\phi_c] \left(\sum_{a\neq b}^{N} \phi_a(x)\ \phi_b(y)\ \right) \exp\left\{-\sum_{b=1}^{N} \mathcal{S}[\phi_b]\right\}}}_{\text{scales with } \propto\ N(N-1)}\right]
\end{aligned} \tag{4.14}$$

[3] Before looking in these terms in equation (4.12) in more detail, it should be understood that the expectation values of the replica fields w.r.t the *theory with replicated fields*, *i.e.* $\mathcal{Z}^N$, are equal to the expectation values of the theory with single scalar field (*without replicas*), independently of the replica indices at the limit $N \to 0$ by construction of the theory

$$\overline{\langle\phi\rangle} = \lim_{N\to 0} \frac{1}{N} \overline{\left\langle \sum_{a=1}^{N} \phi_a\right\rangle_{\mathcal{Z}^N}} = \lim_{N\to 0} \overline{\langle\phi_1\rangle}_{\mathcal{Z}^N} = \lim_{N\to 0} \overline{\langle\phi_2\rangle}_{\mathcal{Z}^N} = \dots = \lim_{N\to 0} \overline{\langle\phi_N\rangle}_{\mathcal{Z}^N}\ . \tag{4.13}$$

This is because all the replica fields are governed by the same statistical distribution. The quantum analogue of this argument would be that the vacuum expectation values of replica fields are identical with the vacuum expectation value of the original theory.

In the second line of the above equality, we can easily see that there are two terms, one which approaches to zero as $N \to 0$ with $\mathcal{O}(N)$ and a term that scales as $\mathcal{O}(1)$ hence non-zero as $N \to 0$. The result for *connected* 2-point correlation function is then

$$\begin{aligned} \lim_{N\to 0} \frac{1}{N} \frac{\delta^2}{\delta J(x)\delta J(y)} \ln\{\overline{\mathcal{Z}^N[\{\phi\},J]}\}|_{J=0} &= \overline{\langle\phi(x)\phi(y)\rangle} - \overline{\langle\phi(x)\rangle\langle\phi(y)\rangle} \\ &= \lim_{m\to 0}\left[\overline{\langle\phi_1(x)\phi_1(y)\rangle}_{\mathcal{Z}^N} - \overline{\langle\phi_1(x)\phi_2(y)\rangle}_{\mathcal{Z}^N}\right] , \end{aligned} \tag{4.15}$$

where in the first line, we have used the previous Eqn. (4.5) for 2-point *connected* correlation function in terms of 1- and 2-point correlation functions. We can now relate the correlation functions in the theory with replicated scalar fields with the correlation functions in the original theory by,

$$\boxed{\begin{aligned} \overline{\langle\phi(x)\phi(y)\rangle} &= \lim_{N\to 0} \overline{\langle\phi_1(x)\phi_1(y)\rangle}_{\mathcal{Z}^N} \\ \overline{\langle\phi(x)\rangle\langle\phi(y)\rangle} &= \lim_{N\to 0} \overline{\langle\phi_1(x)\phi_2(y)\rangle}_{\mathcal{Z}^N} \; . \end{aligned}} \tag{4.16}$$

Hence the *general* 2-point correlation function $\overline{\langle\phi(x)\phi(y)\rangle}$ may be represented as

$$G(x,y) := \overline{\langle\phi(x)\phi(y)\rangle} = \lim_{N\to 0}\frac{1}{N}\left[\frac{1}{\overline{\mathcal{Z}^N}}\overline{\int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\sum_{a=1}^{N}\ \phi_a(x)\ \phi_a(y)\ \exp\left\{-\sum_{b=1}^{N}\mathcal{S}[\phi_b]\right\}}\right] . \tag{4.17}$$

Finally, using relations derived in the previous section, the stochastic average can simply be pushed inside the path integral, simply taking the form,

$$\begin{aligned} &\lim_{N\to 0}\frac{1}{N}\left[\frac{1}{\overline{\mathcal{Z}^N}}\overline{\int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\sum_{a=1}^{N}\ \phi_a(x)\ \phi_a(y)\ \exp\left\{-\sum_{b=1}^{N}\mathcal{S}[\phi_b]\right\}}\right] \\ &\quad = \lim_{m\to 0}\frac{1}{N}\left[\frac{1}{\overline{\mathcal{Z}^N}}\int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\sum_{a=1}^{N}\ \phi_a(x)\ \phi_a(y)\int\mathcal{D}[V]P[V]\exp\left\{-\sum_{b=1}^{N}\mathcal{S}[\phi_b]\right\}\right] , \end{aligned} \tag{4.18}$$

where replacing the relation $\overline{\mathcal{Z}^N} = \int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\exp\{-\mathcal{S}_N\}$ , we arrive at the result

$$G(x,y) := \overline{\langle\phi(x)\phi(y)\rangle} = \lim_{N\to 0}\frac{1}{N}\left[\frac{1}{\overline{\mathcal{Z}^N}}\int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\sum_{a=1}^{N}\ \phi_a(x)\ \phi_a(y)\exp\left\{-\mathcal{S}_N[\{\phi\}]\right\}\right] . \tag{4.19}$$

### 4.1.4 Full Replica Propagator

It is tempting to consider the expression in equation (4.19) through defining a *full* 'replica' propagator in the form of a $N \times N$ matrix,

$$\mathcal{D}_{ab}(x,y) = \frac{1}{\overline{\mathcal{Z}^N}}\int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\ \phi_a(x)\ \phi_b(y)\ \exp\left\{-\mathcal{S}_N[\{\phi\}]\right\} , \tag{4.20}$$

where the stochastic average of the general two-point correlation function becomes

$$\begin{aligned} G(x,y) := \overline{\langle\phi(x)\phi(y)\rangle} &= \lim_{m\to 0}\frac{1}{N}\sum_{a=1}^{N}\mathcal{D}_{aa}(x,y) \\ &= \lim_{N\to 0}\frac{1}{N}\mathrm{Tr}[[\underline{\underline{\mathcal{D}}}(x,y)]] , \end{aligned} \tag{4.21}$$

where we used the notation $\underline{\underline{\mathcal{D}}}$ for the matrix. Moreover, observing equations (4.14) and (4.16), we can also write

$$\overline{\langle\phi(x)\rangle\langle\phi(y)\rangle} = \lim_{N\to 0} \frac{1}{N(N-1)} \sum_{\substack{a,b=1 \\ a\neq b}}^{N} \mathcal{D}_{ab}(x,y). \tag{4.22}$$

The power spectrum in three spatial dimensions is defined as

$$P(k) := \frac{k^3}{2\pi^2} G(k) \ , \tag{4.23}$$

where $k = |\mathbf{k}|$. We assumed power spectrum is independent of time as well as isotropic and homogeneous. In our case this equation will become

$$\boxed{P(k) = \lim_{N\to 0} \frac{k^3}{2N\pi^2} \mathrm{Tr}[[\underline{\underline{\mathcal{D}}}(k)]] \ .} \tag{4.24}$$

## 4.2 Super-horizon dynamics

In this chapter, we will apply the replica formalism to classical super horizon dynamics during inflation. We will begin by calculating corrections to the two-point correlation function for an unperturbed scalar field in de Sitter space with disorder. This approach was first taken in [81–83] where the authors applied these methods to calculate correlation functions for a *test-field* in stochastic inflation setting. Here we extend this analysis by considering a more general inflationary scenario and also metric fluctuations.

### 4.2.1 Physical description

In the case of quenched disorder, we will need a more involved approach to calculate corrections as introduced in the previous sections. The idea of classically sourced disorder on super-horizon scales was introduced in Section §3.4.1. Namely, here we consider the effect from long-wavelength super-horizon modes becoming *quenched* and acting as impurities on the modes around the horizon crossing. Moreover, we will be assuming the modes around the horizon can also be treated classically. Our approach is similar to *stochastic inflation* in formalism while differs considering the physical picture.[4] In what follows, we begin our analysis by first considering a flat FRW background.

### 4.2.2 de-Sitter Space and a Toy Model

In the basic calculations of the previous section we assumed a Minkowski spacetime. In order to promote our formalism into a more cosmological one, let us begin by considering a simple action with a minimally coupled massless scalar field with a kinetic term and a stochastic potential term. Throughout we take Planck mass $M_{Pl} = (8\pi G)^{-1/2}$ and units $c = \hbar = M_{Pl} = 1$ .

$$\mathcal{S}[\phi] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[R - (\partial_\mu\phi)^2 + 2V(\phi;x)\right] \ . \tag{4.25}$$

where we take the metric as

$$g_{\mu\nu} = \mathrm{diag}(1, -a(t), -a(t), -a(t)) \tag{4.26}$$

with $a(t) = \exp(Ht)$ and $\dot{H} \simeq 0$.

[4] In contrast to our approach, in stochastic inflation one considers the coarse-grained effect of sub-horizon quantum fluctuations realised as 'noise' on the dynamics of long-wavelength modes. This also differs considering the treatment of the stochastic effect.

### Replicated action

Following the formalism introduced in the previous sections, we write the action as

$$\mathcal{S}^N[\phi] = \frac{1}{2}\int \mathrm{d}^4x\ a^3(t)\sum_{a=1}^{N}\left[R - (\partial_\mu\phi_a)^2 + 2V(\phi_a;x)\right] \ . \tag{4.27}$$

The stochastic average of the replicated generating functional can be written as

$$\overline{\mathcal{Z}^N} = \int \mathcal{D}[V]P[V]\int\prod_{a=1}^{N}\mathcal{D}[\phi_a]\exp\left\{-\sum_{b=1}^{N}\mathcal{S}^N[\phi_b]\right\} \ , \tag{4.28}$$

where we have once again Wick-rotated to Euclidean action. The next step requires a little more care. We should express the variance of the stochastic potential, i.e $\overline{V^2}$. We begin by making the following assumption: The stochastic potential depends exclusively *only* on time, *i.e.*

$$V(\phi,x)\to V(\phi,t) \ . \tag{4.29}$$

At this point, we will not discuss the time dependence of the correlation function for the random potential and simply write the form in *replica space* as[5]

$$\overline{V(\phi_a,t)V(\phi_b,t')} = \ \mathcal{C}(t,t')\ \mathcal{F}[h_{ab}\phi_a\phi_b] \ , \tag{4.31}$$

where we used a mean field like notation in writing $\phi_a = \phi_a(t,\mathbf{x})$ with $a = \{1,\dots,N\}$ . Moreover, the functional $\mathcal{F}$ (not to be mistaken with free energy $F$) depends on the couplings of the replica fields and a some metric $h_{ab}$ on the scalar replica field space. The generating functional then becomes

$$\begin{aligned}\overline{\mathcal{Z}^N} = \int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\exp\Bigg\{ -\frac{1}{2}\sum_{a=1}^{N}\int \mathrm{d}t\ a^3(t)\int\mathrm{d}^3x\ \left(\ R-(\partial_\mu\phi_a)^2\ \right) \\ +\sum_{a,b=1}^{N}\int\mathrm{d}t\ a^3(t)\int\mathrm{d}t' a^3(t')\int\mathrm{d}^3x\ \mathcal{C}(t,t')\ \mathcal{F}[h_{ab}\phi_a\phi_b]\Bigg\} \ .\end{aligned} \tag{4.32}$$

Note that the replicated potential term is attractive since the all replica fields share the same action. Let us begin simply by assuming short-range temporal correlations

$$\mathcal{C}(t,t')\sim\delta(t-t')\lambda(t) \ . \tag{4.33}$$

Generating functional takes the (compact) form,

$$\overline{\mathcal{Z}^N} = \int\prod_{c=1}^{N}\mathcal{D}[\phi_c]\exp\left\{R-\frac{1}{2}\sum_{a=1}^{N}\int\mathrm{d}t\ a^3\int\mathrm{d}^3x\ \left(-(\partial_\mu\phi_a)^2+\lambda\ a^3\ \sum_{b=1}^{N}\mathcal{F}[h_{ab}\phi_a\phi_b]\right)\right\} \ , \tag{4.34}$$

where we have omitted showing time dependence of the scale factor $a(t) = a$ and $\lambda(t) = \lambda$ . The basic structure of our formalism is essentially the same with that of multi-field inflation [84–86]. We can simply write the scalar part of the action as

$$\boxed{\overline{S^N_\phi} = \frac{1}{2}\sum_{ab}^{N}\int\mathrm{d}t\ a^3\int\mathrm{d}^3x\ \left(-\delta_{ab}(\partial_\mu\phi_a\partial^\mu\phi_b)+\lambda\ a^3\ \mathcal{F}[h_{ab}\phi_a\phi_b]\right) \ .} \tag{4.35}$$

[5]It should be noted, however, that the most widely used definition for the variance of the stochastic potential is

$$\overline{V(\phi)V(\phi')}\propto\delta(\phi-\phi') \ . \tag{4.30}$$

Our formalism intents to be more general in accounting for the full spectrum of couplings among replica fields.

**Background**

Let us calculate the inflationary constraints on the replica potential for this model (*i.e.* $\lambda\mathcal{F}$), assuming spatially flat FRW universe (also see Chapter §2) and homogeneity for the replica scalar fields $\phi_a = \phi_a(t)$ with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{4.36}$$

Scalar replica fields give a set of Klein-Gordon equations akin to that of multi-field scenarios [87]

$$\ddot{\phi}_a + 3H\dot{\phi}_a - \lambda\ a^3\ \delta_{ab}\sum_b \mathcal{F}_{,\phi_b} = 0\ , \tag{4.37}$$

where we simplified our notation with

$$\mathcal{F}[h_{ab}\phi_a\phi_b] \coloneqq \mathcal{F} \quad , \quad \sum_{b=1}^{N} \coloneqq \sum_b \quad , \quad \mathcal{F}_{,\phi_a} = \partial\mathcal{F}/\partial\phi_a \quad , \quad \dot{\phi}_a = \mathrm{d}\phi_a/\mathrm{d}t \quad , \text{ also } \ H = \dot{a}/a\ . \tag{4.38}$$

In order to inflation to happen at all, our scalar potential must be sufficiently flat hence it must satisfy the following conditions,

$$(\mathcal{F}_{,\phi_a})^2 \ll (\mathcal{F})^2 \quad \text{and} \quad \sqrt{\mathcal{F}_{,\phi_a\phi_b}} \ll \mathcal{F}\ . \tag{4.39}$$

Following the above condition, we also assume slow roll dynamics, $\ddot{\phi}_a \ll 3H\dot{\phi}_a$ . Hence in addition to Eqn. (4.39) we have the slow roll condition

$$3H\dot{\phi}_a \simeq \lambda\ a^3\ \delta_{ab}\sum_b \mathcal{F}_{,\phi_a}\ . \tag{4.40}$$

However, we note that the background equations we calculate in this formalism are not physical since we actually are modelling for a single scalar field coupled with gravity driving inflation, *not* multiple fields. Moreover, any measurable physical quantity will be the limiting case where we take the number of replica field to zero $N \to 0$. While our replica fields are not real objects, they are nevertheless realisations of the actual disordered inflationary universe and it is clear that they have to (more or less) satisfy inflationary conditions. Hence in a technical way, they are representative of the 'background distribution'. This is why calculation of the background in this way is not for vain, it constrains our replicated system.

### 4.2.3 Variational Method

Previously we calculated the two-point correlation function as the trace of what we called the *full* replica propagator $\mathcal{D}_{ab}$, which is a $N \times N$ matrix. We will now use the Feynman variational method introduced in Section §3.5 to approximate the *full* replica propagator $\mathcal{D}_{ab}$ with an *a priori* unknown $N \times N$ matrix, $\mathcal{G}_{ab} \to \mathcal{D}_{ab}$, non-perturbatively. We will use the calculations in the previous section to derive replica structure.

**Variational Action**

The machinery is essentially the same as introduced with the exception that we have used a more general formalism here. First, we introduce our variational ansatz in the compact form as

$$\boxed{\mathcal{S}^N_{\text{var}}[\phi] = \int \mathrm{d}t\ a^3 \int \mathrm{d}^3x\ \left(\ \mathcal{G}^{-1}_{ab}\phi^a\phi^b\ \right)\ ,} \tag{4.41}$$

and

$$\boxed{\mathcal{H}^N_{\text{var}} = a^3 \int \mathrm{d}^3x\ \left(\ \mathcal{G}^{-1}_{ab}\phi^a\phi^b\ \right)\ ,} \tag{4.42}$$

where

$$\boxed{\mathcal{G}^{-1}_{ab}(t,\mathbf{k}) = G^{-1}_{0,ab}(t,\mathbf{k}) + \sigma_{ab}\ ,} \tag{4.43}$$

and $G^{-1}_{0,ab} := \delta_{ab} G^{-1}_0(t, \mathbf{k})$ is the propagator for the free part of the action.

### Saddle Point Approximation

Previously we introduced the free energy inequality

$$F \geq \langle \mathcal{S}^N_{\text{var}} - \mathcal{S}^N \rangle_{\mathcal{H}\text{var}} + F_{\text{var}} \ , \tag{4.44}$$

where $F$ is the free energy of the *full* theory and

$$F_{\text{var}} = \ln \mathcal{Z}^N_{\text{var}} \ . \tag{4.45}$$

The expectation value shown with $\langle \cdot \rangle_{\mathcal{H}\text{var}}$ is taken with respect to time-dependent Hamiltonian $\mathcal{H}^N_{\text{var}}$. Our method is to calculate the right-hand side of Eqn. (4.44) and follow with a stationarity equation $\partial F / \partial \mathcal{G}_{ab} = 0$.[6] In order to do this we have to calculate the following expectation value with respect to variational action

$$\int \mathrm{d}^3 x \, \langle \mathcal{F}[h_{ab} \phi_a \phi_b] \rangle_{\mathcal{H}\text{var}} \ . \tag{4.46}$$

We begin by making the observation the for the generic case in Eqn. (4.46), the functional form inside the expectation value can be written as

$$\mathcal{F}\left[ h_{ab} \int_{\mathbf{k}'} \int_{\mathbf{k}} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \phi_{a\mathbf{k}} \phi_{b\mathbf{k}'} \right] \ . \tag{4.47}$$

Performing the spatial integral in Eqn. (4.46) and one of the integrals in phase space in Eqn. (4.47) we make the definition

$$\boxed{\left\langle \mathcal{F}\left[ h_{ab} \int_{\mathbf{k}} \phi_{a\mathbf{k}} \phi_{b-\mathbf{k}} \right] \right\rangle_{\mathcal{H}\text{var}} := \tilde{\mathcal{F}}\left[ \ a^{-3} \int_{\mathbf{k}} h_{ab} \mathcal{G}_{ab} \right]} \ . \tag{4.48}$$

The equations are essentially the same. In order to calculate the replica structure $\sigma_{ab}$ we take the derivate with respect to the propagator $\mathcal{G}_{cd}$. However, the exact result will depend on the form of the functional. The correctness of the saddle point equation relies on two conditions being satisfied, namely the '*free energy*' inequality in Eqn. (4.44) and the stability at the saddle point $\partial F / \partial \mathcal{G}_{ab} = 0$. Next, we will discuss these conditions.

### Saddle point conditions

The inequality in Eqn. (4.44) is satisfied if both sides of the inequality are *convex*. The 'free energy' $F$ is convex by definition since it is a Legendre transform. Nevertheless it is worth noting that we introduced a coarse-grained coupling term between replica fields in the 'effective' action, which mimics the so called 'coarse-grained free energy' in $O(N)$ symmetric field theories studied most extensively in the context of renormalisation group flow in [88–90]. There, authors show that the coarse grained free energy becomes convex for only $k \to 0$ (see *e.g.* [90]). Proving the convexity of the general effective representations in Eqn. (4.44) with this way is beyond the extend of our work here. Instead we will make another observation. Since we take the expectation value with respect to a positive weight (*i.e.* the variational Hamiltonian), we will always *underestimate* the right hand side in Eqn. (4.44). Hence this inequality will necessarily hold. This was first shown by J. Jensen in 1906 [91] and then formalised as the variational principle in Eqn. (4.44) by R. Feynman [80].

On the other hand, the condition $\partial F / \partial \mathcal{G}^\star_{ab} = 0$ is perhaps more stringent and assumes *stationarity* at the point $(t^\star, \mathbf{k}^\star)$. This proplem was first formulated by Thouless et. al. [92]. In our case this

[6] Take care to not mistake the free energy $F_{\text{var}}$ with the replica potential term $\mathcal{F}$ .

corresponds to calculating the eigenvalues of the Hessian of the free energy in replica field space. We denote this as

$$\mathcal{M}_{ab,cd}(t,\mathbf{k}) \propto \frac{\partial^2 \tilde{\mathcal{F}}\left[\int_{\mathbf{k}} h_{ab}\mathcal{G}_{ab}\right]}{\partial \mathcal{G}_{ab}\partial \mathcal{G}_{cd}} \ , \tag{4.49}$$

where we used a compact notation for the functional in Eqn. (4.46). The stability of the saddle-point approximation depends on the positivity of *all* eigenvalues of Eqn. (4.49). It may be very demanding to prove this relation for any given replica potential $\tilde{\mathcal{F}}$. Instead we note that in [79] G. Parisi et. al. showed that for a fully symmetric replica structure (see Section 3.5), this is satisfied for a 3-dimensional manifold if

$$\tilde{\mathcal{F}}\left[\int_{\mathbf{k}} h_{ab}\mathcal{G}_{ab}\right] \propto O(\mathcal{G}_{ab}^{\gamma}) \ , \tag{4.50}$$

with $\gamma \geq 1$. In the rest of this section we will assume this holds in $3+1$ dimensions as well.

### 4.2.4 Variational $\langle\phi\phi\rangle$ calculation

In this section we will calculate the corrections to two-point correlation function of the scalar field $\phi$ from the variational method. Since we are not taking metric fluctuations into account at this point, our method and results are not cosmologically accurate. Our intention at this point is mainly to demonstrate the corrections from various scenarios. In what follows, we will make the simplifying assumption of *full* replica symmetry, *i.e.* $\sigma := \sigma_{ab}$ for $a \neq b$ and $\tilde{\sigma} := \sigma_{aa}$. Moreover we will assume the form for the two point function

$$\langle \phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle^{\star} = G(t^{\star},\mathbf{k}) \simeq \lim_{N\to 0}\frac{1}{N}\mathrm{Tr}[[\mathcal{G}^{\star}_{ab}]] \tag{4.51}$$

where

$$\boxed{\mathcal{G}_{ab}(t,\mathbf{k}) = [\ G_0(t,\mathbf{k})^{-1} + \sigma_{aa}\ ]^{-1}\delta_{ab} + [\ G_0(t,\mathbf{k})^{-1}(\ G_0(t,\mathbf{k})^{-1} + \sigma_{aa})]^{-1}\sigma\ .} \tag{4.52}$$

where the symbol ( $^{\star}$ ) signifies that we have ‘fitted’ $\sigma_{ab}$ to *true* propagator $\mathcal{D}(t,\mathbf{p})$ with the saddle point equation $\partial F/\partial\mathcal{G}_{ab}(t^{\star},\mathbf{k})$ at the given coordinates $\mathbf{k}^{\star}$, $t^{\star}$. The parameter $G_0(t,\mathbf{k})$ is the propagator for the free part of the original action in Eqn. (4.25). In order to calculate the correction from $\sigma^{\star}_{ab}$, we must first make an ansatz on the coupling term in Eqn. (4.31).

#### Monomial Stochastic Potentials

Let us begin by considering the well studied inflationary scenario where one considers a potential in the form

$$V(\phi_a,t) = V_{\delta}\ \phi_a^n \ , \tag{4.53}$$

where $V_\delta$ is some stochastic mass term satisfying $\overline{V_\delta} = 0$ and $\overline{V_\delta^2}(t) \neq 0$ , coupled to some (integer) power of the fields. Once again, the difference in our analysis is that the stochastic term is *quenched*. In order to calculate non-perturbative results, we will now discuss the form of the functional $\tilde{\mathcal{F}}$. Following the notation in Eqn. (4.31), we make the following ansatz

$$\mathcal{F} := \mathcal{F}[(\phi_a\phi_b)^n] \ \text{ and } \ \mathcal{C}(t,t') := \overline{V_\delta^2}(t)\ \delta(t-t') \tag{4.54}$$

where comparing to our previous compact notation, we have $h_{ab} := 1$ and $\lambda(t) := \overline{V_\delta^2}(t)$. We will omit showing the time dependence of the latter in what follows. We begin by noting

$$\tilde{\mathcal{F}} := \tilde{\mathcal{F}}\left[a^{-3}\int_{\mathbf{k}}\mathcal{G}_{ab}\right] \tag{4.55}$$

and

$$\frac{\partial\tilde{\mathcal{F}}}{\partial\mathcal{G}^{\star}_{cd}} = a^{-3}\delta(t-t^{\star})\delta_{ac}\delta_{bd}\times\tilde{\mathcal{F}}'\left[a^{-3}\int_{\mathbf{k}}\mathcal{G}_{ab}\right] \ , \tag{4.56}$$

where we omitted a trivial $\mathbf{p}$-integral in the form $\int_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{k}^\star)$ which comes from the derivative of the argument of the functional

$$\frac{\partial \mathcal{G}_{ab}(t,\mathbf{p})}{\partial \mathcal{G}_{cd}(t^\star,\mathbf{k}^\star)} := \delta(t^\star - t)\delta^3(\mathbf{k}^\star - \mathbf{p})\delta_{ac}\delta_{bd} \ , \tag{4.57}$$

where in what follows we will omit the ( $^\star$ ) symbol in the momentum coordinate $\mathbf{k}$ for simplicity of notation. Moreover, we denote

$$\tilde{\mathcal{F}}'[\chi] := \partial\tilde{\mathcal{F}}[\chi]/\partial\chi \ . \tag{4.58}$$

Applying our formalism derived in the previous chapter, we arrive at the expression for the replica structure

$$\boxed{\sigma^\star_{ab} = a^{\star 3}\lambda^\star \tilde{\mathcal{F}}'\Big[a_\star^{-3}\int_{\mathbf{k}} \mathcal{G}_{ab}\Big] \qquad a \neq b \ .} \tag{4.59}$$

where $a_\star^{-3}$ from Eqn. (4.56) cancels three powers from $a^{\star 6}$ and $a_\star := a^\star$ for clarity of notation. Following our previous arguments, lets first assume the functional $\tilde{\mathcal{F}}$ has the form

$$\begin{aligned} \tilde{\mathcal{F}}[\chi] &\propto \chi^{1+n}/(1+n) \ , \\ \tilde{\mathcal{F}}'[\chi] &\propto \chi^n \ . \end{aligned} \tag{4.60}$$

Using the above relations in Eqn. (4.59), we get

$$\sigma^\star_{ab} = +a^{\star 3}\lambda^\star \left(\frac{a_\star^{-3}}{2\pi^2}\int \mathrm{d}k\ k^2 \mathcal{G}_{ab}(t^\star,k)\right)^n \qquad a \neq b \ , \tag{4.61}$$

where we assumed the propagator for correlations among replica fields depends on the amplitude of the phase vector $|\mathbf{k}|$. Next, we make the observation that in the limit $N \to 0$

$$\lim_{N\to 0} \sigma^\star_{aa} = +a^{\star 3}\lambda^\star \left(\frac{a_\star^{-3}}{2\pi^2 N}\int \mathrm{d}k\ k^2\ \mathrm{Tr}[[\mathcal{G}_{ab}(t^\star,k)]]\right)^n \ , \tag{4.62}$$

and from Eqn. (4.21) we know that the above expression is equivalent to

$$\lim_{N\to 0} \sigma^\star_{aa} = +a^{\star 3}\lambda^\star \left(\frac{a_\star^{-3}}{2\pi^2}\int \mathrm{d}k\ k^2\ G(t^\star,\mathbf{k})\right)^n \ , \tag{4.63}$$

where we see that the contribution from the diagonal elements scale as $\mathcal{O}_{(N)}(1)$ in the limit $N \to 0$ hence we *cannot* ignore them in our calculation. Next, we make the fully replica symmetric anzats, $\sigma_{ab} = \sigma$ where we write

$$\mathcal{G}_{ab}(t^\star,\mathbf{k}) = [\ G_0(t^\star,\mathbf{k}) + \sigma^\star_{aa}\ ]^{-1}\delta_{ab} + [\ G_0(t^\star,\mathbf{k})(\ G_0(t^\star,\mathbf{k}) + \sigma^\star_{aa})]^{-1}\sigma^\star \ . \tag{4.64}$$

Finally by using Eqn. (4.21), we get

$$\begin{aligned} G(t^\star,\mathbf{k}) = \lim_{N\to 0}\frac{1}{N}\mathrm{Tr}[[\mathcal{G}_{ab}(t^\star,\mathbf{k})]] &= \Big[\ G_0(t^\star,\mathbf{k})^{-1} + a^{\star 3}\lambda^\star\tilde{\sigma}^\star\ \Big]^{-1} \\ &\quad + \sigma^\star\Big[\ G_0(t^\star,\mathbf{k})^{-1}\big(\ G_0(t^\star,\mathbf{k})^{-1} + a^{\star 3}\lambda^\star\tilde{\sigma}^\star\big)\Big]^{-1} \ . \end{aligned} \tag{4.65}$$

where

$$\tilde{\sigma}^\star := \left(\frac{a_\star^{-3}}{2\pi^2}\int \mathrm{d}k\ k^2\ G(t^\star,\mathbf{k})\right)^n \ . \tag{4.66}$$

We now return to our initial discussion of the physics that sources the quenched disorder on super-horizon scales. Previously we argued that *super-horizon disorder* effects the modes around the horizon crossing due to longer modes, which have *long passed* the horizon, have become *quenched*. Hence, we expect the faster modes to be effected in a relatively narrow 'window' of momentum around the horizon crossing. In order to realise this, we take our upper limit to be the horizon crossing $k_{\text{h.c.}}$. We write

$$\tilde{\sigma}^{\star} \simeq \left( \frac{a_{\star}^{-3} H^{\star}}{4\pi^2} \int_{\Lambda}^{k_{\text{h.c.}}} \mathrm{d}k \; k^{-1+\epsilon} \right)^n \; , \tag{4.67}$$

where we introduced small deviation parameter from scale invariance $\epsilon := n_s - 1 \ll 1$ (see Section §2.4.3). We observe that this term contributes corrections that increase at most logarithmically in $1/k$ for $\epsilon = 0$ with a high-momentum cut-off at most at the horizon crossing.

### Difference correlation

Replica method was first derived for spin glass models. Spin glass models can crudely be though of disordered Ising models in two-dimensions, where the strength of the couplings between various degrees of freedom is collected from a probability distribution. In replicated spin glass models, each of $N$ realisations is represented by collection of spins situated in an $L \times L$ lattice much like a realisation of various atoms in a crystal. An essential physical parameter, first suggested in [75] and then brought up to sophistication by G. Parisi [93, 94], is the *order parameter*, $\rho_{ab}$ which parametrises a 'distance' measure among realisations in the form $\rho_{ab} \sim \sum_{ij}[\mathbf{s}_a(i) - \mathbf{s}_b(j)]$ where $\{i, j\}$ represents the lattice points of various spin degrees of freedom. From a field theory perspective, this amounts to considering a potential scaling with a 'distance measure' between field realisations, or in other worlds, a *difference correlation* [79].

For our purposes, a difference correlation introduces a power law scaling with respect to the difference between the amplitudes of two realisations of classical super-horizon wave-modes $\phi_a$ and $\phi_b$ at equal time. Let us begin by parametrising the dependence of the functional part of the potential, $\mathcal{F}$, in spatial coordinates. In the simplest form, this object will be as follows,

$$\mathcal{F} := \mathcal{F}[(\phi_a - \phi_b)^2] \; , \quad \text{where} \quad \mathcal{C}(t, t') := \delta(t - t')\lambda(t) \; . \tag{4.68}$$

We then calculate the Fourier transform of this functional and take the expectation value with respect to variational action introduced as before. This results in the functional dependence

$$\tilde{\mathcal{F}} := \tilde{\mathcal{F}}\left[ \int_{\mathbf{k}} (\mathcal{G}_{aa} + \mathcal{G}_{bb} - 2\mathcal{G}_{ab}) \right] . \tag{4.69}$$

and

$$\frac{\partial \tilde{\mathcal{F}}}{\partial \mathcal{G}^{\star}_{cd}} = a^{-3}\delta(t - t^{\star})[\delta_{ac}\delta_{ad} + \delta_{bc}\delta_{bd} - 2\delta_{ac}\delta_{bd}] \times \tilde{\mathcal{F}}' \left[ \int_{\mathbf{k}} (\mathcal{G}_{aa} + \mathcal{G}_{bb} - 2\mathcal{G}_{ab}) \right] . \tag{4.70}$$

Applying the variational formalism as before, we get for the non-diagonal contributions to the replica structure

$$\sigma^{\star}_{ab} = 2a^{\star 3}\lambda^{\star} \left( \frac{1}{2\pi^2} \int \mathrm{d}k \; k^2 (\mathcal{G}_{aa} + \mathcal{G}_{bb} - 2\mathcal{G}_{ab}) \right)^n \quad a \neq b \; , \tag{4.71}$$

where we once again assumed the definition of the functional as in Eqn. (4.60). For the diagonal elements we get

$$\sigma^{\star}_{aa} = - \sum_{a(\neq b)} \sigma_{ab} \; . \tag{4.72}$$

We see that this time in the limit $N \to 0$ the contribution from the diagonal elements can be neglected and the simple form in Eqn. (4.51) holds for the fully symmetric replica structure $\sigma^{\star}_{ab}(k) = \sigma(k)^{\star}$ for $a \neq b$. Moreover by simple arithmetic, we note that for the fully symmetric case we have from Eqn. (3.44) in the limit $N \to 0$,

$$\mathcal{G}_{aa}(k) + \mathcal{G}_{bb}(k) - 2\mathcal{G}_{ab}(k) \simeq G_0(k) \; . \tag{4.73}$$

Finally using the above relations we write

$$\sigma^\star = 2a^{\star 3}\lambda^\star \left( \frac{1}{2\pi^2} \int \mathrm{d}k\ k^2\ G_0(t^\star, \mathbf{k}) \right)^n \ . \tag{4.74}$$

Much like before, this effect can be observed to have at most logarithmically increasing corrections with $1/k$. Moreover since the diagonal contributions (or self-couplings of replica fields) can be neglected in the limit $N \to 0$, replica corrections appear only on the second order in $G_0(t^\star, k)$. Shortly, we find

$$\Delta G(t^\star, \mathbf{k}) \simeq \sigma^\star \times G_0^2(t^\star, k)\ , \tag{4.75}$$

where the fully symmetric replica structure is given in Eqn. (4.74). Hence we see that disorder introduced to single scalar field inflation in this form does not have a *relevant*[7] correction to power spectrum. We will finish this section with few comments.

Our calculations in this section was simply motivated by applying replica formalism and variational method to a *test* field in de Sitter spacetime. In order to calculate any cosmological parameters, we have to sophisticate our formalism. This may be done by considering the metric and scalar field perturbations around a homogenous background. By doing so we could link our corrections to two-point curvature perturbation spectrum which has significance in cosmology. We make an attempt for this in the next section with various simplifying assumptions.

### 4.2.5 Linear perturbation & dynamics

In the previous sections we applied the so called replica field theory method to two simple scenarios. In both, we calculated correlation functions for some classical scalar field in homogeneous de Sitter space described with the metric in Eqn. (4.36). However, if we want to calculate any quantity that has relevance in cosmology, we *must* take into account the *fluctuations* of the metric. In what follows, we will study this using the methods introduced in the review Section §2.4.4.

#### ADM formalism with replica fields

We will introduce perturbation to the metric in Eqn. (4.36). As we discussed previously, there are many ways to do this. We choose to do calculation by following the ADM formalism. We begin by making the observation that the replicated action has the form of an action in Eqn. (4.35) describing multi-field inflation [84, 95] (also see for review *e.g.* [85, 86])

$$\overline{S^N} = \frac{1}{2} \int \mathrm{d}^4x\ \sqrt{-g} \left( R - g^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi^a + \lambda\ a^3\ \sum_{ab} \mathcal{F}[h_{ab}\phi_a\phi_b] \right)\ , \tag{4.76}$$

where we sum over contracted indices. The first term in the brackets is the Einstein tensor, the second term is the canonical kinetic term and the third term is the potential term of the multi-field action. The form of the the action for the perturbed metric in ADM formalism for single field case was given in Eqn. (2.51). In the case of multiple replica fields, the ADM action becomes [84]

$$\begin{aligned} \overline{S^N} = \frac{1}{2} \int \mathrm{d}^4x\sqrt{-g}\Big[ & NR^{(3)} - 2N\lambda\ a^3\ \sum_{ab} \mathcal{F}[h_{ab}\phi_a\phi_b] + N^{-1}(E_{ij}E^{ij} - E^2) - Ng^{ij}\partial_i\phi_a\partial_j\phi^a \\ & + N^{-1}(\dot\phi_a\dot\phi^a - \dot\phi_a(N^i\partial_i\phi^a) - (N^i\partial_i\phi_a)\dot\phi^a + (N^i\partial_i\phi_a)(N^i\partial_i\phi^a))\Big]\ . \end{aligned} \tag{4.77}$$

where we sum over contracted indices. It is important here to note that $g^{ij}$ is the *three-dimensional* metric on slices of *constant* t. Next, the variation of the action with respect to Lagrange multipliers

[7] Here by relevant, we mean correction that scales with $\propto k^{-r}$ where $r \leq 3$.

$N$ and $N_i$ gives the constraint equations (respectively) as in Eqn. (2.53)

$$-g^{ij}\partial_i\phi_a\partial_j\phi^a - 2\lambda\ a^3\ \sum_{ab}\mathcal{F}[h_{ab}\phi_a\phi_b] - N^{-2}(E_{ij}E^{ij} - E^2) + \\ -N^{-2}\Big(\dot{\phi}_a\dot{\phi}^a - \dot{\phi}_a(N^i\partial_i\phi^a) - (N^i\partial_i\phi_a)\dot{\phi}^a + (N^i\partial_i\phi_a)(N^i\partial_i\phi^a)\Big) = 0\ . \tag{4.78}$$

and

$$\nabla_j\left[N^{-1}\big(E^i_j - E\delta^i_j\big)\right] = N^{-1}\dot{\phi}_a\partial_j\phi^a - (N^i\partial_i\phi_a)\partial_j\phi^a\ . \tag{4.79}$$

In order to proceed, we now have to make a gauge choice. Since we are interested at this point with linear perturbations around FRW background, we will use the *flat* gauge

$$g_{ij} := a^2(t)\delta_{ij}\ , \tag{4.80}$$

in order to obtain spatially flat time slices.

**Second order action**

As we discussed in the first chapters, the Universe is observed to be *very* homogenous with only very small deviations from homogeneity. This is realised with the next step in the perturbation theory, which is to split our scalar fields on the flat hyper-surfaces into a homogeneous background component and a part representing the small linear perturbations,

$$\phi^a = \bar{\phi}^a + Q^a\ . \tag{4.81}$$

In order to measure any cosmological observables such as correlation function of curvature perturbations, we need to be able to calculate the fluctuations $Q$ around the homogeneous background. Here, we will describe the procedure that follows Eqn. (4.81) and will not show all the steps in calculation. We begin by redefining the Lagrange multipliers as before, $N = 1 + \delta N$ and $N_i = \partial_i\chi$ which become [84]

$$\delta N = \frac{1}{2H}\dot{\bar{\phi}}_a Q^a\ , \tag{4.82}$$

and

$$\nabla^2\chi = \frac{a^2}{2H}\nabla^2\left(Q_a\ddot{\bar{\phi}}^a - \dot{\bar{\phi}}_a\dot{Q}^a - \frac{\dot{H}}{H}\dot{\bar{\phi}}_a Q^a\right)\ . \tag{4.83}$$

In order to write the action, one expands Eqn. (4.35) up to quadratic order in terms of linear perturbations. Substituting the expressions in Eqn. (4.82) and Eqn. (4.83), one arrives at an expression of the action for the scalar perturbations. Here we give the final result from [84]

$$\boxed{\overline{S_2^N}[Q] = \frac{1}{2}\int \mathrm{d}t\ \mathrm{d}^3x\ a^3\left(\partial_\mu Q_a\partial^\mu Q^a - \lambda\ a^3\ \mathcal{F}_{;ab}Q^aQ^b + \mathcal{J}_{ab}\dot{Q}^a\dot{Q}^b\right)\ .} \tag{4.84}$$

where $\mathcal{F}_{;ab} := \partial^2\mathcal{F}/\partial\phi_a\partial\phi_b$ and $\mathcal{J}_{ab}$ is defined as

$$\mathcal{J}_{ab} := \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{H}\dot{\bar{\phi}}_a\dot{\bar{\phi}}_b\right)\ . \tag{4.85}$$

In what follows we will assume Eqn. (4.85) is negligible.

### 4.2.6 Variational $\langle QQ\rangle$ corrections

First we redefine our variational action and Hamiltonian for fluctuations as follows

$$\boxed{\mathcal{S}^N_{\mathrm{var}} = \int \mathrm{d}t\ a^3\int \mathrm{d}^3x\ \left(\ \mathcal{G}^{-1}_{ab}Q^aQ^b\ \right)\ ,} \tag{4.86}$$

$$\boxed{\mathcal{H}^N_{\text{var}}(t) = a^3 \int \mathrm{d}^3x \ \left( \ \mathcal{G}^{-1}_{ab} Q^a Q^b \ \right) \ ,} \tag{4.87}$$

where in what follows, we will assume the free energy inequality in Eqn. (4.44) holds for Eqn. (4.84) and (4.86).[8]

### Perturbative disorder

The replica field action for linear fluctuations $\{Q_a\}$ around a flat FRW background satisfied by all realisations of the replica fields $\{\phi_a\}$ in Eqn. (4.84) is a general representation of the most disorder scenarios.[9] In what follows, however, we will make the simplifying assumption that the disorder is *perturbative*, namely the homogeneous background part of the replica field is same for each realisation

$$\phi^a(t, \mathbf{k}) \simeq \bar{\phi}(t) + Q^a(t, \mathbf{k}) \ . \tag{4.88}$$

### Difference correlation revisited

The general action given in Eqn. (4.84) may take a complicated form depending on the replica field couplings realised by the functional $\mathcal{F}$. Here we will make the simplifying anzats where we assume *difference correlation* among replica fields,

$$\mathcal{F} := \mathcal{F}[(\phi_a - \phi_b)^2] \simeq \mathcal{F}[(Q_a - Q_b)^2] \ . \tag{4.89}$$

where in the second line we used the fact that we take disorder to have a perturbative effect. Moreover, observe that the second derivative of the functional can be written as,

$$\mathcal{F}_{;ab} = \sum_{n=1}^{\infty} \frac{\mathcal{F}^{(n)}}{(n-1)!} \left( (n-1) \left( \bar{\phi}^2 + \bar{\phi}(Q_a + Q_b) - 2(Q_a - Q_b)^2 \right) - 2(Q_a - Q_b)^2 \right) (Q_a - Q_b)^{2n-2} \ . \tag{4.90}$$

As can be seen, the above expression is constant, $\mathcal{F}_{;ab} \propto \mathcal{O}(1)$ for $n := \{0, 1\}$. Hence we find that the lowest non zero contribution from the second term in Eqn. (4.84) is quadratic in replica fields. By making the definition

$$\tilde{\mathcal{F}}_\delta := \int \mathrm{d}^3x \, \langle Q_a Q_b \mathcal{F}_{;ab} \rangle_{\text{var}} \tag{4.91}$$

we see for the term in Eqn. (4.90) of first order $\mathcal{O}(1)$ in replica field fluctuations, this gives

$$\tilde{\mathcal{F}}_\delta \underbrace{=}_{\text{LO}} a^{-3} \int_{\mathbf{k}} \mathcal{G}_{ab} \ , \tag{4.92}$$

and

$$\sigma^\star := \sigma^\star_{ab} = a^{\star 3} \lambda^\star \quad \text{for all } \{a, b\} \ . \tag{4.93}$$

This would give an constant additive correction to the two point correlation function which is otherwise free from disorder

$$\langle Q_{\mathbf{k}} Q_{-\mathbf{k}} \rangle^\star = G^\star(k) \simeq \left[ G^\star_0(k)^{-1} + \sigma^\star \right]^{-1} + \sigma^\star \left[ G^\star_0(k)^{-1} (G^\star_0(k)^{-1} + \sigma^\star) \right]^{-1} \ , \tag{4.94}$$

where $G^\star_0(k) = H^\star / 2k^3$. Moreover since the fluctuations are perturbative for any given realisation, $\bar{\phi} \gg Q_a$, we can generalise this equation for any order in replica field fluctuations in the coupling term

[8]For this statement to be general, we may have to require for the fluctuations to decouple in Eqn. (4.84) from the homogeneous background evolution for a given replica field $\bar{\phi}_a$. In the next section, however, we will assume perturbative disorder where this will not be an issue.

[9]To be exact, we should clarify that the action in Eqn. (4.84) applies to scenarios where disorder does *not* effect the kinetic term in Eqn. (4.76).

by making the simplifying approximation,

$$\mathcal{F}_{;ab} \simeq \bar{\phi}^2 \sum_{n=2}^{\infty} \frac{\mathcal{F}^{(n)}}{(n-2)!} (Q_a - Q_b)^{2n-2} \underbrace{=}_{n\geq 2} \bar{\phi}^2 \mathcal{F}[(Q_a - Q_b)^2] \,. \tag{4.95}$$

### 4.2.7 Disordered kinetic coupling

In the previous sections we introduced replica field approach with variational method in a simple way, considering a limited range of scenarios. We have realised disorder as a stochastic mass-like term coupled to a scalar field. We then applied the replica trick and calculated correlation functions by coupling realisations where we used a saddle-point approximation to get the most suitable correction to the free propagator. Our variational approach forced us to calculate a constant replica structure $\sigma_{ab}$ which we found to have no significant effect on the power spectrum. This is since different phases must *decouple* in the action so that we can calculate the variational expectation $\langle \cdot \rangle_{\mathcal{H}\text{var}}$ for a given term quadratic in replica fields. It is tempting, however, to ask within this simple formalism, whether one might consider a way to have a correction to the correlation functions that scales with some power of $\mathbf{k}$.[10] One way to consider such a scenario is to introduce disorder as a stochastic potential coupled to a *kinetic* term. Here, we will consider the following coupling

$$\mathcal{S}^N = \frac{1}{2}\sum_{ab}^{N} \int \mathrm{d}t \; \lambda \; a^6 \int \mathrm{d}^3x \; \mathcal{F}\left[h_{ab}\frac{1}{a^2}\partial_i\phi_a\partial^i\phi_b\right] \,, \tag{4.96}$$

where $i := \{1,2,3\}$ spatial coordinates. Variational expectation value then returns

$$\langle \mathcal{S}^N \rangle_{\mathcal{H}\text{var}} = \frac{1}{2}\sum_{ab}^{N} \int \mathrm{d}t \; \lambda \; a^6 \; \mathcal{F}\left[a^{-5}\int_{\mathbf{p}} |\mathbf{p}|^2 h_{ab}\mathcal{G}_{ab}\right] \,. \tag{4.97}$$

For the action in Eqn. (4.35) we get (for $h_{ab} := 1$)

$$\boxed{\sigma^\star_{ab}(k) = a^\star\lambda^\star k^2 \; \tilde{\mathcal{F}}'\left[\frac{a_\star^{-5}}{2\pi^2}\int \mathrm{d}p \; p^4 \; \mathcal{G}^\star_{ab}(p)\right] \quad \text{for all} \quad a,b \,.} \tag{4.98}$$

where we used the saddle point equation $\partial F/\partial\mathcal{G}_{ab}(t^\star,\mathbf{k}) := 0$. Moreover, the diagonal terms are

$$\boxed{\lim_{N\to 0}\sigma^\star_{aa}(k) = a^\star\lambda^\star k^2 \; \tilde{\mathcal{F}}'\left[\frac{a_\star^{-5}}{2N\pi^2}\int \mathrm{d}p \; p^4 \; \mathrm{Tr}[[\mathcal{G}^\star_{ab}(p)]]\right] \,.} \tag{4.99}$$

where we have used the relations $k = |\mathbf{k}|$ and $p = |\mathbf{p}|$ as well as assuming the replica propagator depends only on the magnitude of the vector $\mathbf{p}$. Moreover in Eqn. (4.99) we used the relation

$$\mathcal{G}^\star_{aa} = \frac{1}{N}\sum_a \mathcal{G}^\star_{aa} = \frac{1}{N}\mathrm{Tr}[[\mathcal{G}^\star_{ab}]] \,, \tag{4.100}$$

which is true if all diagonal elements in the replica structure are the same $\tilde{\sigma} = \sigma_{aa}$, or true in general for $N \to 0$. Finally using the relation in Eqn. (4.52) and Eqn. (4.51) we can write the replica propagator $\mathcal{G}_{ab}$ as

$$\boxed{G^\star(k) = \Big( \; G_0^\star(k)^{-1} + k^2 \; \tilde{\sigma}^\star\Big)^{-1} + k^2 \; \tilde{\sigma} \; \Big( \; G_0^\star(k)^{-1}\Big( \; G_0^\star(k)^{-1} + k^2 \; \tilde{\sigma} \; \Big)\Big)^{-1},} \tag{4.101}$$

[10] Here we will make a clarification. This goal can not be achieved by introducing some spatial dependence on the variance of the stochastic potential since this would reflect only as a constant correction to the correlation function. One alternative approach would be to directly promote the replica structure to scale with momentum $\sigma \to \sigma(\mathbf{k}) := \xi(\mathbf{k})\sigma$ where *e.g.* $\xi(\mathbf{k}) \propto k^3$ and we fit for $\sigma$ only. However we will be considering a more physical reason for the momentum dependence.

where

$$\tilde{\sigma}^\star := a^\star \lambda^\star \tilde{\mathcal{F}}' \left[ \frac{a_\star^{-5}}{2\pi^2} \int \mathrm{d}p \; p^4 \; G^\star(p) \right] . \tag{4.102}$$

The results in Eqn. (4.101) show the $\mathbf{k}$ dependent correction to two-point function in the presence of superhorion disorder effecting the kinetic term in a scalar field action in de Sitter spacetime.

However, so far for the kinetic consideration we have only derived some basic equations for a scalar field in de-Sitter spacetime. In order for our calculations to represent the physics of inflation, we *must* consider metric fluctuations and introduce disorder perturbatively as previously studied. We will not attempt this as this would require a multi-field perturbation theory with a non-canonical kinetic term as studied in [86]. Instead in the next section we will consider an alternative scenario, EFT of inflation.

### 4.2.8 *No* super-horizon treatment of EFT of Inflation?

We begin with the stating that the EFT of inflation formalism introduced in Section §3.4.2 is appropriate for micro-scale physics of sub-horizon fluctuations.[11] Our discussion, on the other hand, involve very long super-horizon wave modes that are frozen and have become classical which we associate as the source of disorder. Even if we assume these modes have a stochastic effect on the faster modes represented in the EFT formalism, we would still need to calculate correlation functions around (if not outside) the horizon crossing where the gauge invariance, *i.e.* the $\mathcal{R} \sim -H\pi$ correspondence, is no longer valid. This is simply because the Goldstone modes $\pi$ continue to grow *outside* the horizon. Nevertheless, we will still make an attempt to justify this section by considering a window of validity in which one can calculate the correlation functions on super-horizon scales, classically. This would be *after* the horizon crossing to allow Goldstone modes to have become classical, and *before* too much after the horizon crossing in order for the formalism to be still applicable. Moreover, in what follows, we will introduce disorder to the effective action in Eqn. (2.80) as a perturbative effect generating non-trivial spatial kinetic coupling of replica fields in the form

$$\overline{V(\pi_a, t)V(\pi_b, t')} = M_{\mathrm{dis}}^4(t) \; \delta(t - t') \left(\partial_i \pi_a \partial^i \pi_b\right) . \tag{4.103}$$

where we consider the following effective action

$$\boxed{\mathcal{S}_{\mathrm{eff}}^N = \int \mathrm{d}t \; a^3 \int \mathrm{d}^3x \left( -M_{\mathrm{pl}}^2 \dot{H} \left( \dot{\pi}_a \dot{\pi}^a - \frac{\partial_i \pi_a \partial_i \pi^a}{a^2} \right) + a^3 M_{\mathrm{dis}}^4 \sum_{ab} \frac{\partial_i \pi_a \partial^i \pi_b}{a^2} \right) .} \tag{4.104}$$

By using the above relation in Eqn. (4.101) with $\tilde{\sigma}^\star = a^\star M_{\mathrm{dis}}^{\star 4}$ we find

$$\langle \pi_{\mathbf{k}} \pi_{-\mathbf{k}} \rangle = \left( G_0^\star(k)^{-1} + k^2 a^\star M_{\mathrm{dis}}^{\star 4} \right)^{-1} + k^2 a^\star M_{\mathrm{dis}}^{\star 4} \left( G_0^\star(k)^{-1} \left( G_0^\star(k)^{-1} + k^2 a^\star M_{\mathrm{dis}}^{\star 4} \right) \right)^{-1} . \tag{4.105}$$

where in this case

$$G_0^\star(k)^{-1} := \frac{H_\star^2}{2M_{\mathrm{pl}}^2 |\dot{H}_\star|} \frac{1}{k^3} . \tag{4.106}$$

#### The speed of sound $c_s$

When introducing a term like in Eqn. (4.103), it might be interesting to consider the diagonal replica ‘self-interaction’ terms for the spatial kinetic term which take the form

$$\left(-M_{\mathrm{pl}}^2 \dot{H} + a^3 M_{\mathrm{dis}}^4\right) \frac{\partial_i \pi_a \partial^i \pi_a}{a^2} . \tag{4.107}$$

[11] We would like to make clear for the reader, by EFT of inflation, we refer to the EFT describing microscopic physics of inflation. Not the superhorizon EFT (see *e.g.* [96]) in stochastic inflation theories, which is most possibly another reasonable setting for such classical treatment.

Since the spatial and time kinetic terms are different, introducing disorder in this way will 'change' the speed of sound $c_s$ for the Goldstone modes $\pi$ that is otherwise $c_s = 1$. This is can be defined as

$$c_{ab}^2 := \delta_{ab} - \frac{a^3 M_{\text{dis}}^4}{M_{\text{pl}}^2 |\dot{H}|} \ . \tag{4.108}$$

Using this relation, we can rewrite the action as

$$\boxed{\mathcal{S}_{\text{eff}}^N = \int \mathrm{d}t\ a^3 \int \mathrm{d}^3x \left( -M_{\text{pl}}^2 \dot{H} \left( \dot{\pi}_a \dot{\pi}^a - c_{ab}^2 \frac{\partial_i \pi^a \partial_i \pi^b}{a^2} \right) \right) \ .} \tag{4.109}$$

Finally we now relate the two-point function for Goldstone modes to the two-point correlation function for curvature perturbations around the horizon crossing under the influence of disorder. We do this by taking into account the change in speed of sound (for the free propagator) by rescaling the momentum $k \to c_s k$ where we defined $c_s := \frac{1}{N}\text{Tr}[[c_{ab}]]$. We find

$$\boxed{\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{-\mathbf{k}} \rangle = \left( \tilde{G}_0^{-1}(k) + k^2 a^\star M_{\text{dis}}^{\star 4} \right)^{-1} + k^2 a^\star M_{\text{dis}}^{\star 4}\ \tilde{G}_0^\star(k) \left( k^2 a^\star M_{\text{dis}}^{\star 4} + \tilde{G}_0^{-1}(k) \right)^{-1} \ ,} \tag{4.110}$$

where

$$\tilde{G}_0^\star(k) := \left( \frac{H_\star^4}{2M_{\text{pl}}^2 |\dot{H}_\star|} \frac{1}{c_s^{\star 3} k^3} \right) \ . \tag{4.111}$$

In Figure 4.1 we plot the deviation from the free power spectrum as a function of speed of sound in the large $k$ limit which scales cubically in the leading order. By expanding the relation in Eqn. (4.110) around a small deviation from $c_s \sim 1$ we get

$$\boxed{\tilde{\Delta}_{\mathcal{R}}^2(\tilde{k}) \simeq 1 + \left( -3 + \frac{H_\star^3}{a^\star\ \tilde{k}} + \mathcal{O}[H_\star^6] \right) \lambda_{\text{dis}} + \left( 3 - \frac{3H_\star^3}{a^\star \tilde{k}} - \frac{H_\star^5}{a_\star^2 \tilde{k}^3} + \mathcal{O}[H_\star^6] \right) \lambda_{\text{dis}}^2 + \dots \ ,} \tag{4.112}$$

where

$$\tilde{\Delta}_{\mathcal{R}}^2(k) := \tilde{k}^3\ \tilde{P}(\tilde{k}) \ , \tag{4.113}$$

and

$$\tilde{k} := k/H \quad , \quad \tilde{P}(k) := P(k)/H \quad , \quad 2M_{\text{pl}} |\dot{H}| \simeq 1 \quad , \quad \lambda_{\text{dis}} := 1 - c_s^2 \ . \tag{4.114}$$

Moreover we normalised this expression with $2M_{\text{pl}}^2 |\dot{H}| \sim 1$ and used the relation $M_{\text{pl}}^2 |\dot{H}| \gg H^4$ to collect terms higher order in $H$ in Eqn. (4.112). As we have seen before in Chapter §2, the disorder-free analogue of the expression in Eqn. (4.112) is scale invariant. We see that the kinetic coupling of disorder introduces a *small* constant correction along with scale-dependent corrections higher order in $H$. It is apparent from Eqn. (4.112) that the deviation from scale invariance in higher orders in $H$ is a small and effect since our effective formalism is valid only around the horizon crossing. It might nevertheless be interesting to calculate the deviation from scale invariance in Eqn. (4.112) with more care. We leave this for future work. We will conclude this section with a discussion on the *constant* part of the correction.

### 4.2.9 Discussion

In calculating classical corrections to correlation functions, we have so far limited ourselves to introducing replica method and variational approach in a simple and general way. The corrections we calculated were scale invariant with the exception of Sections §4.2.7 and §4.2.8. In these two sections we discussed the scenario where the kinetic term is influenced by disorder and found that the replica structure $\sigma(k)$ scales with momentum squared $k^2$. Nevertheless, it is apparent from our results that disorder introduced in this way is irrelevant as $k \to \infty$. However, since we are considering the effect of

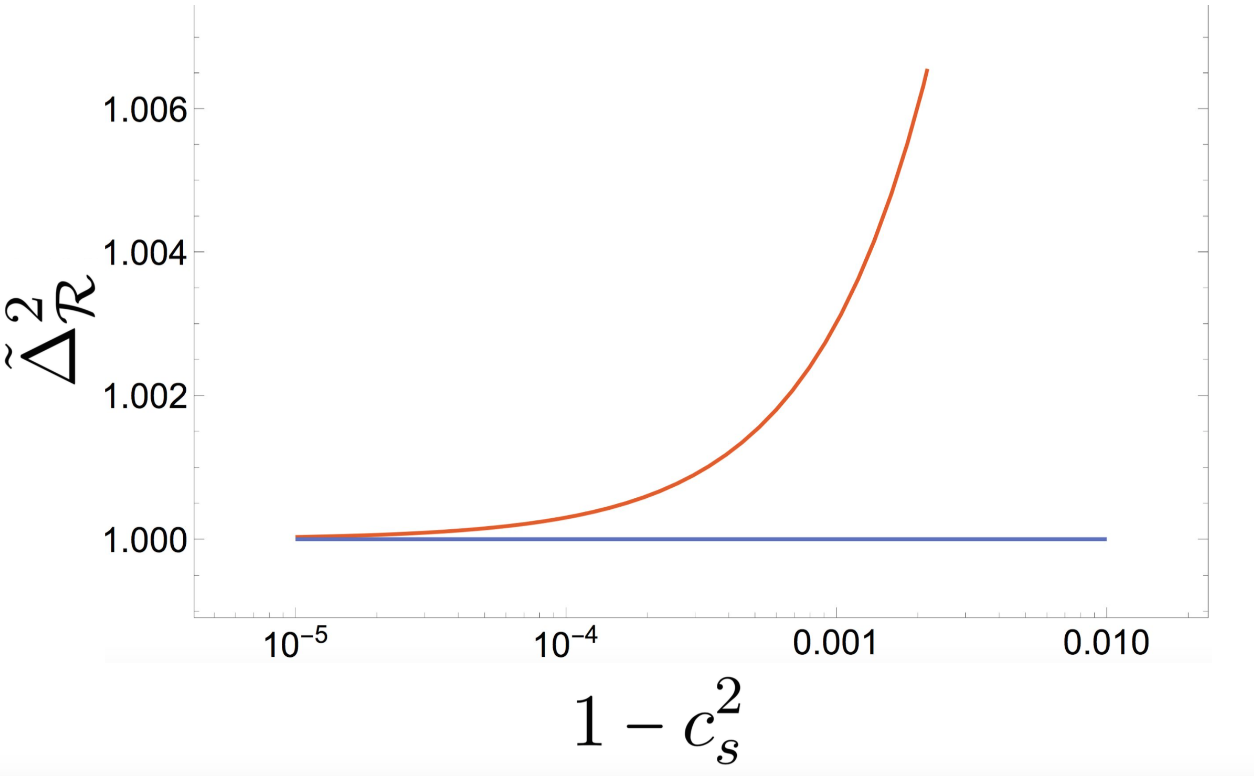


Figure 4.1: The normalised scale independent power spectrum as function of speed of sound $c_s$. The normalised parameter in vertical axis is scaled to satisfy the relation $\Delta^2_{\mathcal{R}} \propto \tilde{k}^3\ \tilde{P}_{\mathcal{R}}(\tilde{k})$ where $\tilde{k} := k/H$ and $\tilde{P}_{\mathcal{R}}(\tilde{k}) := P_{\mathcal{R}}(\tilde{k})/H$. Finally in this plot we attained $2M_{\rm pl}^2|\dot{H}| \sim 1$ and used the relation discussed in Section §2.5, $M_{\rm pl}^2|\dot{H}| \gg H^4$. The orange line represents the scenario with kinetic disorder while the blue line represents the case without of disorder. This plot simply shows that the deviation from the free case increases cubically with decreasing speed of sound $c_s$.

disorder to influence the modes around the horizon crossing, it is worth discussing if one might have large disorder corrections in this region. Let us begin by considering the corrections from the replica structure calculated in Eqn. (4.65) and (4.66)

$$\tilde{\sigma}^\star := \left( \frac{a_\star^{-3}}{2\pi^2} \int_{\Lambda_{\rm IR}}^{\Lambda_2} \mathrm{d}k\ k^2\ G(t^\star, \mathbf{k}) \right) , \tag{4.115}$$

where we have replaced the bounds of the phase integral with cut-offs with an low-momentum IR-cut-off, $\Lambda_{\rm IR}$, and some high momentum cut-off $\Lambda_2$. For a scale invariant spectrum, the above equation gives a logarithmically divergent correction term in the form of

$$\tilde{\sigma}^\star = \left( \frac{a_\star^{-3}}{2\pi^2} \ln \left\{ \frac{\Lambda_2}{\Lambda_{\rm IR}} \right\} \right) . \tag{4.116}$$

We will now consider the high-momentum cut-off. This has to be at most at horizon crossing $\Lambda_2 := k_{\rm h.c.} \sim a^\star H^\star$ to allow for the wave-modes to have chance to become *frozen* and contribute to disorder. We note that the replica structure scales with $\tilde{\sigma}^\star = a_\star^{-3}\ \ln\{a^\star\}$, hence we see that $\tilde{\sigma} \to 0$ as $a_\star \to 0$. Now lets consider a replica structure in the form Eqn. (4.102)

$$\tilde{\sigma}^\star := a^\star \lambda^\star \tilde{\mathcal{F}}' \left[ \frac{a_\star^{-5}}{2\pi^2} \int \mathrm{d}k\ k^4\ G^\star(k) \right] . \tag{4.117}$$

If we make the definition for the functional, $\tilde{\mathcal{F}}'[\chi] := \chi^n$, we get

$$\tilde{\sigma}^\star := a^\star \lambda^\star \left( \frac{a_\star^{-5}\ g_\star\ k_{\text{h.c.}}^2}{2\pi^2} \right)^n \tag{4.118}$$

where we used $G^\star(k) := g_\star k^{-3}$. The factor $k_{\text{h.c.}}$ is the momentum amplitude at the horizon crossing. It is clear from the above expression that the effect of disorder rapidly vanishes for $a_\star \to \infty$ and positive $n$. We find that disorder introduced to super-horizon scales in this section is likely to remain very small and vanish with $a \to 0$ and $k \to 0$.

We complete this section with few remarks. The way we introduced disorder into the super-horizon description of inflation was highly limited by our ability to decouple different phase modes in variational method. This constrained us to consider only a few types of couplings and we only studied short-range correlations between coordinates. The next step towards a more complete theory would be to better define the variational method. We will leave this for future work. The application of the replica method, however, is quite general and can be treated as a multi-field scenario with the total number of fields taken to a limit at the end of calculations. The calculations that survive this analytic continuation contribute to the *physical* parameters of the system. In fact, we have explicitly seen this when calculating the replica structure throughout this section.[12]

## 4.3 Towards a quantum theory of early Universe disorder

Much like in the super-horizon scales, a common consensus on cosmological quantum theory of sub-horizon quenched *disorder* does not exist. Our intention in this chapter is to review the formalisms with which one can study the microscopic disorder in the early Universe. In what follows, we will begin by considering perturbative treatment of disorder. We will then extend our understanding of cosmological *in-in* formalism and relate it to the non-equilibrium quantum field theories that have much relevance to the study of disorder in condensed matter scenarious. We will conclude this section with a discussion on possible extension of the formalism introduced so far with the replica theory and furthermore emphasise the similarities between the super-horizon classical framework introduced in Section §4.2. We have discussed the analogies between disorder in condensed matter physics and various phenomena in early Universe cosmology in Sections §3.3 and §3.4. In contrast to our classical super-horizon calculations, the quantum treatment of sub-horizon dynamics is much richer with emergent phenomena.[13] Perhaps the consideration of quenched disorder effects are most appropriate shortly after inflation, during (p)reheating since it is phenomenologically suggestive that the physics is much less constrained with symmetries and highly non-adiabatic during this period. While in the following sections we focus on inflationary dynamics, our discussion and formalism applies for post-inflationary dynamics as well.

### 4.3.1 Perturbative treatment of disorder

In Section §4.1 we introduced the replica method in calculating correlation functions. The necessity of replicating the system was discussed in previous sections and directly related to quenched characteristic of the disorder. We will now return to this issue and discuss it in dept. In the presence of quenched disorder one *cannot* take an average over disorder in the same footing as the system due to the difference between the time-frames of the impurities and system's own degrees of freedom. From a phenomenological perspective, this is related to *non-ergodicity* of the system, where due to quenched nature of the impurity, an ensemble average of the local degrees of freedom in a given volume will

[12] It is suggestive that within this formalism, the replica structure, which mimics the self energy for the system, shows universality as it does not depend strongly on the number of replica realisations.

[13] For clarity, we point out that *complexity* is not necessarily a quantum phenomena. Nevertheless, we make this statement since we are interested in quantum phenomena such as particle production and localisation due to interference, etc.

not be equal to the asymptotic values of the systems' parameters reach in time. In other words, a system with *quenched disorder* is in a *non-equilibrium* state. This brings us to the issue of making measurements on a non-equilibrium system by simply *perturbing* it. By definition, if a system is in a non-equilibrium state, even a very small perturbation, which would otherwise be followed by system's rapid conversion to its true ground energy state, may have much longer lasting effects on the system. As we discussed before, this was why we applied the variational method as a non-perturbative approach to calculate quenched disorder effects on super-horizon scales in Section §4.2. The replica structure gave us the ability to *probe* the *free energy landscape* and calculate for the minimum free energy of the given system. We now wish to do the same for a sub-horizon theory of disorder. In Section §4.2, the simplicity of our calculations relied on the classical nature of the fields and the theory. In order to consider non-equilibrium systems for quantum theories, we have to sophisticate our formalism significantly. We will make this attempt in what follows. First, however, we will consider the simplistic scenario where we *can* treat disorder perturbatively for illustrative purposes.

#### Annealed disorder

We will begin by considering a scenario where the disorder *can* be treated with perturbation theory. This means that our system is more or less in equilibrium and shows ergodicity. Hence, here we will be treating disorder essentially as it is *annealed* disorder (see Section §3.2.1). Although *annealed* disorder is phenomenologically less rich than *quenched* disorder, it is still interesting as it can introduce *noise* to the power spectrum as well as resonant features which we will discuss shortly. Since the quantum theory of disorder is much more involved than its classical analogue, the *annealed* disorder case would be a good starting point to gain some insight into the calculations. The first approach on this was taken recently by D. Green in [97]. In what follows, we will reviewing the calculations done in [97]. There, the authors refer to the disorder as *quenched* while the perturbative treatment in the rest of the work necessarily degrades the disorder to *annealed* characteristic instead. Nevertheless, authors describe the quenched characteristic as the lack of *feedback* between the field fluctuation and the perturbatively introduced stochastic coupling term. By this, they refer to the fact that the stochastic potential they introduce does not *exclusively* depend on spatial coordinates, hence it induces no dissipative effect which would reflect back to the background equations of the system. This is much like the way *we* introduced classical disorder in the previous section with similar motivations. Moreover, this simple assumption allowed us to easily decouple different modes from each other and perform the expectation value with respect to a variational Hamiltonian. However, this has no direct relation our definition of the *quenched* characteristic of the disorder. Next, we will review the results in [97].

#### EFT of inflation formalism

Throughout this chapter and we will be mainly using the EFT of inflation formalism as it provides a simple and general treatment of inflationary microscopic dynamics.[14] We already introduced the effective field theory formalism for inflation in Eqn. (2.80). Here, we begin by introducing perturbative disorder with the split

$$M_{\rm pl}^2\dot{H}(t) \to M_{\rm pl}^2\dot{H}(t) + M_{\rm pl}^2\dot{h}(t) \quad \text{and} \quad M_{2,3}^4(t) \to M_{2,3}^4(t) + m_{2,3}^4(t)\ , \tag{4.119}$$

where the terms $\dot{H}$ and $M_{2,3}^4$ are fixed while $\dot{h}$ and $m_{2,3}$ are Gaussian stochastic variables. Next, we will give the resulting corrections on the $n$-point correlation functions calculated in [97]. We introduced the *in-in* formalism for calculating correlation functions in Section §2.8.1. Introducing the *annealed* stochastic effect to our action, the resulting *perturbative* 2nd order correction to $n$-point correlation

[14] Also see Section §3.4.2 for further justification when making analogies between inflation and condensed matter physics.

becomes

$$\overline{\left\langle Q(\tau)^{(2)}\right\rangle} = \sum_{i,j}\left(\int\limits_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\tau_1 a^4(\tau_1) \int\limits_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\tau_2 a^4(\tau_2) \left\langle \mathcal{O}_i(\tau_1) Q(\tau) \mathcal{O}_j(\tau_2)\right\rangle \mathcal{C}_{ij}(\tau_1,\tau_2) \right.$$
$$\left. - 2\mathrm{Re} \int\limits_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\tau_1 a^4(\tau') \int\limits_{-\infty(1-i\epsilon)}^{\tau_1} \mathrm{d}\tau'' a^4(\tau_2) \left\langle \mathcal{O}_i(\tau_1) \mathcal{O}_j(\tau_2) Q(\tau)\right\rangle \mathcal{C}_{ij}(\tau_1,\tau_2)\right) , \tag{4.120}$$

where

$$\mathcal{C}_{ij}(\tau_1,\tau_2) := \overline{(x_i(\tau_1)x_j(\tau_2))} \quad , \quad x_{i,j} = \{M_{\mathrm{pl}}^2 \dot{h}, m_{2,3}^4\} \quad \text{and} \quad \mathcal{O}_{i,j} = \{\dot{\pi}^2, -a^{-2}\partial_i \pi \partial^i \pi\} \; . \tag{4.121}$$

Throughout this chapter we will use conformal time $\tau \sim -\frac{1}{aH}$. Moreover we assume that the fields satisfy $\pi_{\mathbf{k}} := \bar{\pi}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} + \mathrm{h.c.}$ and

$$\bar{\pi}_{\mathbf{k}} = \frac{H}{2M_{\mathrm{pl}}|\dot{H}|^{1/2}} \frac{1}{k^{3/2}} (1 - ik\tau) e^{ik\tau} \; , \tag{4.122}$$

where $\hat{a}^\dagger_{\mathbf{k}}$ is the creation operator. Finally we choose the variance of the stochastic potential term to be [97]

$$\mathcal{C}_{ij}(\tau,\tau') = \delta_{ij} \frac{M_{\mathrm{pl}}^4 |\dot{H}|^2}{\Lambda_i} (-H\tau)^{p+1} \delta(\tau - \tau') \; , \tag{4.123}$$

where $p$ allows some deviation from scale invariance. The calculations are straight forward and we will omit the details and directly give the results and follow with a discussion.

### Disorder corrections from [97]

We will consider corrections from the $\dot{h}$ term. This can be considered as corrections to slow roll inflation. The calculation is done by placing the operators $\mathcal{O}_1 := \dot{\pi}^2$, $\mathcal{O}_2 := a^{-2}\partial_i \pi \partial^i \pi$ and $x_{\{1,2\}} := M_{\mathrm{pl}}^2 \dot{h}$ into Eqn. (4.120). One arrives at

$$\Delta \left\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{-\mathbf{k}} \right\rangle_{\mathrm{dis}} = -\frac{H}{\Lambda_1} P_{\mathcal{R}}(k) \frac{4 + (1-p)p}{2-p} \cos\left(\frac{p\pi}{2}\right) \Gamma[p] \left(\frac{H}{2k}\right)^p \tag{4.124}$$

where in the physical limit $p \to 0$ this becomes

$$\lim_{p\to 0} \Delta \left\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{-\mathbf{k}} \right\rangle_{\mathrm{dis}} = -2\frac{H}{\Lambda_1} P_{\mathcal{R}}(k) \left[\frac{1}{p} + \frac{3}{4} - \left(\gamma + \ln\left\{\frac{k}{2H}\right\}\right)\right] \; . \tag{4.125}$$

We used the notation $\langle \cdot \rangle_{\mathrm{dis}}$ to indicate the averaging over disorder. It is apparent that this correction is not entirely physical since it diverges in the limit $p \to 0$ and moreover the scale invariance is violated by the finite part where we see a multiplicative logarithmic correction. We will discuss these issues in the next section. Here, we also give the results for the *trispectrum* corrections in slow-roll as given in [97]

$$\lim_{p\to 0} \Delta \left\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \mathcal{R}_{\mathbf{k}_4} \right\rangle'_{\mathrm{dis}} = \frac{H}{64\Lambda_1} P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_3) \delta(\mathbf{k}_1 + \mathbf{k}_2) \delta(\mathbf{k}_3 + \mathbf{k}_4) \times$$
$$\left(\underbrace{\frac{-16k_1^2 + 31k_1k_3 - 16k_3^2}{(k_1 - k_3)^2}}_{\text{resonant excitations}} + \underbrace{\frac{16(k_1^2 + k_3^2)}{k_1 k_3} \mathrm{ArcTanh}\left(\frac{k_3}{k_1}\right)}_{\text{varying amplitude}}\right) + \mathrm{perm.} \tag{4.126}$$

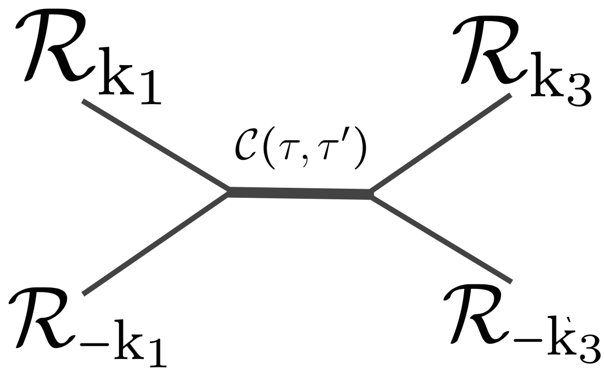


Figure 4.2: Diagrammatical demonstration of the trispectrum in Eqn. (4.126) where the exchange between two pairs of fields is due to the stochastic effect.

### Resonance excitations and further discussion

In the previous section we have written the divergent contributions from disorder for both the power spectrum and trispectrum calculated in [97]. This divergence is due to the delta-function in Eqn. (4.123) where we defined the variance of the stochastic potential. The delta-function allows for arbitrarily rapid shifts in the stochastic parameters. Stochastic parameters determine the energy at which the states can be excited, which in return are allowed to become arbitrarily large. This is best understood with an analogy of 'resonance' which have been discussed in [97–100]. We can realise this by relating the disorder to oscillatory features. We do this by writing

$$x_i = \int \frac{\mathrm{d}\omega}{2\pi} \lambda_\omega e^{i\omega t} \ , \tag{4.127}$$

where $\omega$ is the comoving frequency and

$$\mathcal{C}(\omega,\omega') = \frac{M_{\mathrm{pl}}^4 \dot{H}^2}{\Lambda_1}(2\pi)\delta(\omega+\omega') \ , \tag{4.128}$$

which gives Eqn. (4.123) in the limit $p \to 0$. From this equation it is apparent that disorder introduces arbitrary fluctuations to the system. One way to compensate for this is to introduce an exponential suppression beyond some *frequency* cut-off for $\omega > \Lambda_L$ where the delta function is not a good approximation. For the two-point calculation, this results in including a factor $e^{\epsilon_L k\tau}$ where $\epsilon_L := H/\Lambda_L$ in the path integral. With this suppression, the power spectrum in Eqn. (4.125) becomes

$$\lim_{p\to 0} \Delta \langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{-\mathbf{k}} \rangle = \frac{1}{2}\frac{H}{\Lambda_1} P_{\mathcal{R}}(k) \left[ -3 - 2\ln\left\{ \frac{\epsilon_L^2}{4} \right\} \right] \ . \tag{4.129}$$

For the 4-point function, we should consider suppression terms in the form $\exp\left\{\frac{1}{2}\epsilon_L\tau_1 \sum_i k_i\right\}$ where $k_i$ represents the external momenta. We will conclude this section with further discussion on the the corrections to the trispectrum in Eqn. (4.126). We have by definition forced the stochastic coupling potential to *not* depend on the spatial coordinates *exclusively*. This resulted in *no*-momentum exchange between the two pairs of fields in trispectrum representation. Nevertheless the trispectrum is irreducible since the time dependence of the stochastic potential leads to exchange of energy. We illustrated this diagramatically in Figure 4.2 where the variance of the stochastic term $\mathcal{C}(\tau,\tau')$ realises the exchange between the field pairs. Finally the second term in large brackets in Eqn. (4.126) represents the effect of disorder in randomly varying the power spectrum.

### Adding disorder to inflationary models

Although the perturbative characteristic of the disorder in [97] differs from our discussion in the previous sections, the formalism introduced there is nevertheless informative. Specifically from an effective field theory perspective, the general framework of disorder allows one to consider a wide spectrum of possible applications. In fact, the calculations reviewed in this section suggest the possibility of simply *adding* perturbative disorder to inflationary models. On this note, we return shortly to trapped inflation [69] introduced in Section §3.4.3. If we consider the effect of disorder as it randomises the locations where a particle $\chi_i$ is produced, this would promote the particle number density to a stochastic quantity. Let us consider the total number density of produced particles in the form $n(t) \sim \sum_i n_{\chi_i}(t)$. With the effect of disorder, this quantity is to be collected from a probability distribution $P(n;t)$. In order to calculate the evolution of the total number density of produced particles $n$, one could then formulate a Fokker-Planck equation for the probability distribution $P(n;t)$. Similar procedure have been considered for non-adiabatic particle production during reheating in [60]. It is indeed suggestive that the inflationary models such as trapped inflation may prove fruitful in considering quenched disorder.

## 4.3.2 *In*-*In* formalism revisited

In this section we will review some of the initial steps towards going beyond the perturbation theory and studying a larger spectrum of phenomena in relation to disorder in the early Universe. We begin by returning to our discussion of the *in-in* formalism.

### Review

In Section §2.8.1 we introduced the *in-in* formalism and have given the 'master formula' in Eqn. (2.97). There, we used the operator formalism for expectation values which simplified considerably in calculating corrections to correlation functions perturbatively. In order to gain more insight into the theory, let us start by considering a diagrammatical formalism. We will now discuss the Feynman rules for the *in-in* formalism. We consider drawing all diagrams for the calculation of the expectation value for a given term $\langle Q \rangle$ of $N$th order in interaction. Quite different from the *in-out* formalism, this time all vertices are distinguished as '$R$' and '$L$' due to the difference between the time-ordered and anti-time-ordered product. Namely, the '$R$' label represents the field propagating forward in time and label '$L$' represents field propagating backwards in time. We show the path integral in Figure 4.3. For a diagram with $N$ vertices, there will be $2^N$ ways one can choose each vertex to be either '$R$' or '$L$' which would contribute either $-i$ or $+i$ respectively. Moreover the formalism gets even more complicated as we consider the lines connecting these vertices to each other. We have three types of exchange between vertices. A line connecting a two right vertices (or a right vertex to an *external* line) contributes the Feynman propagator $\langle \mathrm{T}\{\phi(x)\phi(x')\}\rangle$ where as a line connecting two left vertices contributes a propagator in the form $\langle \bar{\mathrm{T}}\{\phi(x)\phi(x')\}\rangle$. Finally, we will have lines connecting a left (right) vertex to a right (left) vertex which contribute the propagator $\langle \phi(x)\phi(x')\rangle$. With this better understanding of the *in-in* formalism, we will now review the path integral.

### The *in-in* path integral

In order to go beyond perturbation theory, it is much more preferable to use the path-integral formalism rather than the operator formalism in Eqn. (2.97). We begin by writing the path integral as given in [20]

$$
\begin{aligned}
\langle Q(t)\rangle = &\int \prod_{t',n} \frac{\mathrm{d}\xi_{Ln}(t')}{\sqrt{2\pi}} \prod_{t',n} \frac{\mathrm{d}\xi_{Rn}(t')}{\sqrt{2\pi}} \\
&\times \exp\left\{-i\int_{-\infty}^{t} \mathrm{d}t'\ \mathcal{L}[\xi_L(t'),\dot{\xi}_L(t')';t]\right\} \exp\left\{i\int_{-\infty}^{t} \mathrm{d}t'\ \mathcal{L}[\xi_R(t'),\dot{\xi}_R(t')';t]\right\} \\
&\times \left(\prod_n \delta(\xi_{Ln}(t)-\xi_{Rn}(t))\right)\ Q(\xi_L(t))\ \Psi_0^*(\xi_L(-\infty))\Psi_0(\xi_R(-\infty))\ ,
\end{aligned} \tag{4.130}
$$

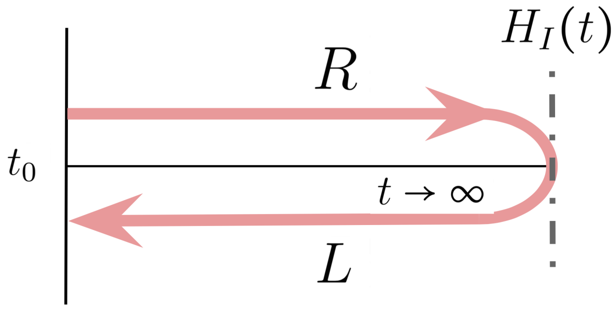


Figure 4.3: Demonstration of the *in-in* formalism closed time path (CTP) integral..

where the field $\xi$ represents field *fluctuations* around a homogeneous background and the index $n$ includes all fields and their conjugates as well as spatial coordinates. The *paths* within the exponential values are written as

$$\mathcal{L}[\xi(t'),\dot{\xi}(t')';t] := \sum_a \int \mathrm{d}^3x \; \Pi_a(t',\mathbf{x})\dot{Q}_a(t',\mathbf{x}) - \mathcal{H}[Q(t'),\Pi(t');t'] \;, \tag{4.131}$$

where we represent the conjugate fields as $\Pi$. Index $a$ runs from all the fields realised in the Lagrangian. The variable $\Psi_0[\xi]$ in Eqn. (4.130) is the *vacuum wave function* taken at the initial time

$$\begin{aligned}\Psi_0[\xi(-\infty)] &\propto \langle in|\xi(-\infty)\rangle\,\langle\xi(-\infty)|in\rangle \\ &= \exp\left(-\frac{\epsilon}{2}\int_{-\infty}^{t}\mathrm{d}t' e^{\epsilon t'}\sum_{ab}\int \mathrm{d}^3x\int \mathrm{d}^3y\;\mathcal{E}_{ab}(\mathbf{x},\mathbf{y})Q_a(t')Q_b(t')\right)\end{aligned} \tag{4.132}$$

where $\epsilon$ is a positive, infinitesimal, exponentially damping factor used to make the path integral *converge* once Wick rotated to Euclidean signature[15]. The parameter $\mathcal{E}_{ab}$ is a positive-definite kernel.[16] Next, we expand the Lagrangian $\mathcal{L}$ into a quadratic term $\mathcal{L}_0$ and an interaction term $-\mathcal{H}_I$. The quadratic term can be written as

$$\mathcal{L}_0[\xi(t'),\dot{\xi}(t')';t] := \sum_a \int \mathrm{d}^3x \; \Pi_a(t',\mathbf{x})\dot{Q}_a(t',\mathbf{x}) - \mathcal{H}_0[Q(t'),\Pi(t');t'] \;, \tag{4.134}$$

Together with the vacuum wavefunction, we can now define the free propagators of the system as

$$\begin{aligned}\frac{1}{2}\sum \mathcal{G}^R_{0,nn'}(t',t'')\xi_{Rn}(t')\xi_{Rn'}(t'') := &\int_{-\infty}^{t}\mathrm{d}t'\Big\{\mathcal{L}_0[\xi_R(t'),\dot{\xi}_R(t');t'] \\ &+\frac{i\epsilon}{2}\sum_{ab}\int \mathrm{d}^3x\int \mathrm{d}^3y\;\mathcal{E}_{ab}(\mathbf{x},\mathbf{y})Q_{Ra}(\mathbf{x},t)Q_{Rb}(\mathbf{y},t')\Big\}\end{aligned} \tag{4.135}$$

and similarly

$$\begin{aligned}\frac{1}{2}\sum \mathcal{G}^L_{0,nn'}(t',t'')\xi_{Ln}(t')\xi_{Ln'}(t'') := &\int_{-\infty}^{t}\mathrm{d}t'\Big\{\mathcal{L}_0[\xi_L(t'),\dot{\xi}_L(t');t'] \\ &-\frac{i\epsilon}{2}\sum_{ab}\int \mathrm{d}^3x\int \mathrm{d}^3y\;\mathcal{E}_{ab}(\mathbf{x},\mathbf{y})Q_{La}(\mathbf{x},t)Q_{Lb}(\mathbf{y},t')\Big\},\end{aligned} \tag{4.136}$$

[15] This is often called the '$i\epsilon$' prescription and explained in detail in [12].

[16] For a real scalar field, this is simply

$$\mathcal{E}(\mathrm{x},\mathrm{y}) := \frac{1}{(2\pi)^3}\int \mathrm{d}^3p e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\sqrt{\mathbf{p}^2+m^2}\;. \tag{4.133}$$

note however the sign difference between two expressions. Next, one writes the product of delta functions in Eqn. (4.130) in terms of a Gaussian distribution

$$\begin{aligned}\prod_n \delta(\xi_{Ln}(t) - \xi_{Rn}(t)) &\propto \exp\left(-\frac{1}{\epsilon'}\sum_n (\xi_{Ln}(t) - \xi_{Rn}(t))\right) \\ &= \exp\left(-\sum_{nn'} \mathcal{C}_{nn'}(t',t'')\big(\xi_{Ln}(t') - \xi_{Rn}(t')\big)\big(\xi_{Ln}(t'') - \xi_{Rn}(t'')\big)\right)\end{aligned} \tag{4.137}$$

where

$$\mathcal{C}_{nn'}(t',t'') := \frac{1}{\epsilon'}\delta_{nn'}\delta(t'-t)\delta(t''-t) \ , \tag{4.138}$$

and $\epsilon'$ is yet *another* positive infinitesimal. The delta function serves as the integral over the so-called *boundary* field which contracts the '$R$' and '$L$' fields in the *free* theory. In addition to the *free* part of path integral, the interaction Hamiltonian introduces (from a diagrammatical language) lines that connect right and left *vertices* to each other, *e.g.* $iG^{RL}_{nn'}(t',t'')$ and also to themselves, *e.g.* $-iG^{RR}_{nn'}(t',t'')$. The $G$'s are determined by the condition [20]

$$\begin{pmatrix} iG_0^R - \mathcal{C} & \mathcal{C} \\ \mathcal{C} & -iG_0^L - \mathcal{C}\end{pmatrix}\begin{pmatrix} -iG^{RR} & iG^{RL} \\ iG^{LR} & iG^{LL}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \tag{4.139}$$

where we have written $G^{LR} := (G^{RL})^{\mathrm{T}}$. Because the above equality must be independent of the exact value $\epsilon'$, we get the following relations

$$G_0^R\ G^{RR} = 1 \ , \quad G_0^L\ G^{LL} = 1 \ , \tag{4.140}$$

$$G_0^R\ G^{RL} = 0 \ , \quad G_0^L\ G^{LR} = 0 \ , \tag{4.141}$$

$$\mathcal{C}G^{LL} = \mathcal{C}G^{RL} \ , \quad \mathcal{C}G^{RR} = \mathcal{C}G^{LR} \ , \tag{4.142}$$

The first and second equations in Eqn. (4.140) are the inhomogeneous wave equation for the propagator and its complex conjugate with solutions

$$G^{RR}_{nn;}(t',t'') = i\,\langle \mathrm{T}\{\xi_n(t')\xi_{n'}(t'')\}\rangle \ , \tag{4.143}$$

and

$$G^{LL}_{nn'}(t',t'') = -i\,\langle \bar{\mathrm{T}}\{\xi_n(t')\xi_{n'}(t'')\}\rangle \ , \tag{4.144}$$

where we the $n$ and $n'$ indices include all fluctuation fields and conjugates and their spatial coordinates. Moreover, from Eqn. (4.141) we see that $G^{RL}$ and $G^{LR}$ satisfy the *homogeneous* analogue of wave equations satisfied by $G^{RR}$ and $G^{LL}$. Finally using Eqn. (4.142) and Eqn (4.138) we find

$$iG^{RL}_{nm}(t_1,t_2) = \langle \xi_m(t_2)\xi_n(t_1)\rangle \ , \tag{4.145}$$

and

$$iG^{LR}_{nm}(t_1,t_2) = \langle \xi_m(t_1)\xi_n(t_2)\rangle \ . \tag{4.146}$$

We will conclude our review of the *in-in* formalism with a discussion of the generating functional.[17] Differently from the *in-out* formalism first introduced in Eqn. (4.2), we now have with boundaries at the initial time and interaction at some return time $t$. Hence, we can write the generating functional for the *in-in* formalism as

$$\mathcal{Z}\left[J_R, J_L\right] = \int \mathcal{D}\phi\,\langle in|\phi,t\rangle_{J_R}\langle \phi,t|in\rangle_{J_L} \ , \tag{4.147}$$

[17] See for further discussion *e.g.* [101].

where the integral of the *internal* part is the identity operator. This is essentially the path integral given in Eqn. (4.130) where we now have two source terms $J^R$ and $J^L$ representing the independent 'right' and 'left' external sources with which the transition amplitudes given in Eqn. (4.147) are calculated.

### 4.3.3 Non-equilibrium (quantum) field theories

Throughout this work we have discussed the non-equilibrium scenario as the natural environment to study *quenched* disorder. We will now take a look into state-of-the-art quantum non-equilibrium field theories, see *e.g.* [62]. Here, we intend to point the reader towards future work in this area in relation to what has been discussed in this paper. We first review the generating functional and the path integral in non-equilibrium field theories.

#### The generating functional

We begin by noting that all aspects of the *in-in* formalism introduced so far applies directly to non-equilibrium field theories and have central role.[18] Much like in cosmology, for non-ergodic systems, the only boundary conditions that is known about the system and can be applied are at the initial time. Naturally in the non-equilibrium field theories, one also uses the closed time path integral given in Figure 4.3. The generating functional for correlation functions in non-equilibrium theory is given as

$$\begin{aligned} Z[J_{L/R}, R; \rho_0] = \mathrm{Tr}\Bigg\{ \rho_0 \; \mathrm{T}_{\mathcal{C}} \exp\Big\{ \, i\big(\mathcal{S}[\phi] + \int \mathrm{d}^4x \; J_R(x)\phi_R(x) - \int \mathrm{d}^4x \; J_L(x)\phi_L(x) \\ + \frac{1}{2}\int \mathrm{d}^4x \; \mathrm{d}^4y \phi(x) R(x,y)\phi(y)\big)\Big\}\Bigg\} \; , \end{aligned} \tag{4.148}$$

where in order to make the inversely directed fields apparent we have written the external source terms in a compact notation. The only difference this time is the third term in the exponential, which is $R(x,y)$.[19] We will study this term shortly. First, lets further understand the formalism in its correspondence to the *in-in* prescription. We begin by calculating connected two-point functions which are essentially the $G$ propagators we calculated previously

$$\frac{\delta^2 \mathcal{Z}[J_{L/R}, R]}{i\delta J_R(x) i\delta J_R(y)}\Big|_{J,R=0} \coloneqq G^{RR}(x,y) + \phi(x)\phi(y) \; , \tag{4.150}$$

$$\frac{\delta^2 \mathcal{Z}[J_{L/R}, R]}{i\delta J_L(x) i\delta J_L(y)}\Big|_{J,R=0} \coloneqq G^{LL}(x,y) + \phi(x)\phi(y) \; , \tag{4.151}$$

$$\frac{\delta^2 \mathcal{Z}[J_{L/R}, R]}{i\delta J_R(x) i\delta J_L(y)}\Big|_{J,R=0} \coloneqq G^{RL}(x,y) + \phi(x)\phi(y) \; , \tag{4.152}$$

$$\frac{\delta^2 \mathcal{Z}[J_{L/R}, R]}{i\delta J_L(x) i\delta J_R(y)}\Big|_{J,R=0} \coloneqq G^{LR}(x,y) + \phi(x)\phi(y) \; . \tag{4.153}$$

where we area taken the initial conditions to be Gaussian, *i.e.* $\mathrm{Tr}[\rho_0] \to 1$. The expressions for $G$ propagators are same as introduced in the previous section.

[18] In non-equilibrium field theories of condensed matter physics, this formalism is called Schwinger-Keldysh.

[19] One may also find the term $\rho_0$ unfamiliar. This term simply parametrises the initial conditions, namely it is

$$\mathrm{Tr}[\rho_0] \coloneqq \int \mathcal{D}\phi_L \mathcal{D}\phi_R \Psi_0^\star(\phi_L(-\infty))\Psi_0(\phi_R(-\infty)) \; . \tag{4.149}$$

For the rest of the paper we will assume the initial conditions are Gaussian $\mathrm{Tr}[\rho_0] \to 1$.

### Effective action

We now review the effective action in non-equilibrium quantum field theory. We begin by defining logarithm of the partition function $\mathcal{W}$, *i.e.* the 'free energy' same as in Eqn. (4.3) with the partition function $\mathcal{Z}[J_{L/R}, R]$ having different source terms. A Legendre transform with respect to these source terms will contribute different correlation functions. More specifically, a Legendre transform with respect to source term $J_{L/R}$ will give one particle irreducible *effective action* shown with $\Gamma[\phi]$. This object is parametrised only by the one-point function $\phi$. A Legendre transform with respect the bilinear source term $R$ in Eqn. (4.148) leads to the two-particle irreducible (2PI) effective action $\Gamma[\phi, G]$ parametrised by the two-point function. These are

$$\frac{\delta \mathcal{W}[J_{L/R}, R]}{\delta J_a(x)} := \langle \phi_a(x) \rangle \ , \tag{4.154}$$

and

$$\frac{\delta \mathcal{W}[J_{L/R}, R]}{\delta R_{ab}(x,y)} := \frac{1}{2} \left( \phi_a(x)\phi_b(y) + G_{ab}(x,y) \right) \ . \tag{4.155}$$

where for simplifying of the notation we omit writing the terms 'L' and 'R' for the one-point and two-point functions. In the above equation and for the rest of this section, we will consider all one point functions (*e.g.* $\phi_a$) to carry either 'R' or 'L' depending on the source term as well as every two-point function (*e.g.* $G_{ab}$) to carry two indices with either 'R' or 'L' each as shown in Equations (4.150) to (4.153). In order to get the *standard* 1PI effective action $\Gamma[\phi]$ from Eqn. (4.154), one sets $R = 0$. For a more general theory with non-zero R, one promotes the standard expressions to correspond for the bilinear source term. This amounts to writing [43]

$$G_0^{-1}(\phi) \rightarrow G_0^{-1}(\phi) - iR \ , \tag{4.156}$$

where we have used matrix notation for the free propagator $G^{-1}_{0,ab}$ and the source term $R_{ab}$. Much like we have done before in the classical application of variational method, one approximates the 1PI effective action with a saddle-point approximation to find

$$\Gamma^R[\phi]_{(1)} = S^R[\phi] + \frac{i}{2} \mathrm{Tr} \ln[G_0^{-1}(\phi) - iR] \ , \tag{4.157}$$

where $S^R[\phi]$ is the modified *classical action* given as

$$S^R[\phi] := S[\phi] + \frac{1}{2} \int R_{ab}(x,y)\phi_a(x)\phi_b(y) \ . \tag{4.158}$$

In what follows we will take a closer look into the 2PI effective action, also in relation to our previous calculations. The 2PI effective action is a functional of the fields $\phi_a(x)$ and the propagator $G_{ab}(x,y)$ as given in Eqn. (4.155). One calculates this by writing

$$\Gamma[\phi, G] = W[J_{L/R}, R] - \int \phi_a(x) J_a^{R/L}(x) - \frac{1}{2} \int \left( \phi_a(x)\phi_b(y) + G_{ab}(x,y) \right) R_{ab}(x,y) \ . \tag{4.159}$$

Note here the stationarity condition is given as

$$\frac{\delta \Gamma[\phi, G]}{\delta G_{ab}(x,y)} = -\frac{1}{2} R_{ab}(x,y) \ , \tag{4.160}$$

which is the equation of motion for the propagator $G_{ab}$. For the 1 loop expression of 2PI effective action gives

$$\Gamma[\phi, G]_{(1)} = \Gamma^R[\phi]_{(1)} - \frac{1}{2} \int [\phi_a(x)\phi_b(y) + G_{ab}(x,y)] R_{ab}(x,y) \ . \tag{4.161}$$

Using this equation and also the stationarity condition in Eqn. (4.160) we arrive at the relation

$$\frac{\delta\Gamma[\phi,G]_{(1)}}{\delta G_{ab}(x,y)} = -\frac{i}{2}G^{-1}_{ab}(x,y) + \frac{i}{2}G^{-1}_{0,ab}(x,y;\phi) = -\frac{1}{2}R_{ab}(x,y) \ . \tag{4.162}$$

or equivalently

$$G^{-1} = G_0^{-1}(\phi) - iR \ . \tag{4.163}$$

Now we write these relations back into the general equation in Eqn. (4.159) to find[20]

$$\boxed{\Gamma[\phi,G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\ G_0^{-1}(\phi)G + \Gamma[\phi,G]_{(2)} + \mathrm{const.}} \tag{4.164}$$

where the second from last term represents the higher order loop expansions of the effective action. Finally we will make an attempt to better understand the term $\Gamma[\phi,G]_{(2)}$. Applying a variational approach to the expression in Eqn. (4.164) we rewrite the propagator as

$$\boxed{G^{-1}_{ab}(x,y) = G_0^{-1}(x,y;\phi) - iR_{ab}(x,y) - \Sigma_{ab}(x,y;\phi,G) \ .} \tag{4.165}$$

where

$$\Sigma_{ab}(x,y;\phi,G) := 2i\frac{\delta\Gamma_2[\phi,G]}{\delta G_{ab}(x,y)} \ . \tag{4.166}$$

The propagator $\Sigma_{ab}$ is the *proper* self-energy which satisfy the relation first given in Eqn. (3.40) where we have promoted the free propagator to $G_0^{-1} \to G_0^{-1} - iR$. Next, we will conclude this section with a discussion on these relations.

**Discussion**

By considering the *effective action* formalism in quantum field theory in Section §4.3.3, we have written the expression for the *proper* self energy, $\Sigma_{ab}$, which is a functional of the coordinates as well as the fields. The general equation in Eqn. (4.164) is calculated following the same prescription as we had in our classical calculations, *i.e.* the saddle-point approximation for the effective action and stationary condition with respect to the *full* propagator of the system. In fact perhaps it is reasonable to consider the replica approach with variational method studied in Section §4.2 as a simpler classical analogue of the formalism introduced in this section. In Section §4.2 we begin with a single field action with a stochastic mass-like term. We achieved the bilinear representation of field couplings by replicating the system to $N$-realisations and we formulated an action for all $N$-realisations. At this point the $N$ component action is completely analogous to a multi-field action representation with arbitrary couplings between fields as we have discussed in §4.2.5. The classical variational method introduced in Section §4.2.3 allowed us to parametrise the contribution from the field couplings which corresponds to the non-diagonal part of the replica 'structure' $\sigma_{ab}$. As discussed before, $\sigma_{ab}$ mimicks the self-energy $\Sigma$. Of course the action in Eqn. (4.164) does not reflect $N$ realisations. In order to incorporate the 'replica' fields into this action, one only needs to add an additional index that runs through all replicas. This is not uncommon in contemporary disordered condensed matter physics non-equilibrium quantum field theory studies (see *e.g.* [102]) and allows for a rich set of possibilities where one can map the $N$-replica system into some other formalism and calculate the effects from replica field couplings before taking $N \to 0$ or 1 depending on the details of the problem. Finally, considering the formalisms introduced throughout this paper, we arrive at the understanding that a natural quantum treatment of *quenched* disorder in early Universe cosmology may be through studying non-equilibrium variational effective action in the *in-in* (or Schwinger-Keldysh) formalism.

[20]See also for review *e.g.* [43].

# Chapter 5

# Conclusion

Perhaps one of the more notable discoveries in theoretical physics within the last decades has been the significance of the *dualities* between formalisms. From a phenomenological perspective, a duality could correspond to *mapping* a dynamic system, which may be well understood in one branch of physics, onto another. In this work, we tried to take a step in this direction and study the condensed matter phenomena in early Universe cosmology. We have mainly focused on the mechanism of *disorder* as studied in condensed matter physics. Studying the *quenched* character of this effect lead us to consider alternative methods than what is currently available to early Universe cosmology. In the last chapter, we made attempts towards establishing formalisms for calculating the effects of such phenomena. We will now summarise our ideas.

In Chapter §3 we introduced various analogies between condensed matter physics and early Universe cosmology. There, we had two main arguments. First, we considered inflation as a condensed matter phenomena and made analogies between the propagation of a particle in an environment, such as in a condensed matter system, and the propagation of the inflaton through it's potential. In this discussion, we focused mainly on effective field theory (EFT) of inflation which has additional properties that further helped with our analogy. These properties include the decoupling from gravity and exact correspondence of inflaton fluctuations with curvature perturbations at the horizon crossing. Here, we make the observation that the EFT of inflation is indeed a suitable formalism for establishing links with condensed matter phenomena. This has significance for the discussion in this paper and also for the future study we suggest in this area. Our second main argument in Chapter §3 was to introduce the phenomenological correspondence between the early Universe and the condensed matter systems in extreme conditions. We collected these under the definition of 'non-equilibrium' systems. There, we reviewed thermalisation, prethermalisation and related phenomena in relation to inflation and post-inflationary dynamics.

From a more technical point of view, our main focus in Chapter §3 and the following chapter is *disorder*. We started discussing disorder as it was discovered in mid-late 20th century. This also allowed us to introduce the elementary notions of *complexity* and *emergence*. These have significance in our study and constitute to our main motivation in our consideration of disorder. Especially, we studied the quenched disorder as the facilitator of many emergent phenomena including localisation and percolation which we discussed in Chapter §3. Next in the same chapter, we reviewed some of the contemporary efforts in the modern condensed matter research. These consisted mainly of studying non-equilibrium systems' dynamics which we also related to quenched disorder. One particular phenomena we mentioned was the emergence of slow dynamics. Particularly from a phenomenological perspective, it is suggestive that the emergence of slow dynamics may have a wide application in early Universe cosmology. With this, and also other reasons established in this chapter, we understand that a better understanding of disorder is important in considering some of the interesting phenomena in the early Universe. Finally in this chapter, we introduced a replica field theory formalism for calculating correlation functions in disordered systems.

At the end of Chapter §3, we proceeded to suggest that the cosmology lacks an adequate formalism to study quenched disorder during inflation. Hence in Chapter §4, we suggested 'replica field theory' as a formalism to study classical disorder effects in super-horizon scales. This formalism allows us to probe the minimum free energy landscape and estimate the self-energy in a classical field theory. We applied this to inflation and calculated corrections to power spectrum. While at the first order these corrections are irrelevant, it might be interesting to see if this formalism allows us to have physical signatures that can be observable. We leave this analysis for future work. Next in Chapter §4, we made an attempt to discuss the quantum field theory of disorder in the early Universe. There, we made two important observations. First, we suggested the possibility of adding perturbative disorder to inflationary models. We followed a previous attempt in developing a formalism for perturbative disorder in effective field theories of inflation. Considering disorder as a perturbative effect on various inflationary models is possibly an interesting approach and may turn out to be fruitful given the wide range phenomena related to disorder. Trapped inflation is perhaps a good place to start considering such effects. We also leave this to future work. Our second observation considers specifically the *quenched* disorder. Following our discussion in Chapter §4 on the fact that the quenched disorder necessarily demands a non-perturbative treatment, we made an attempt to consider a quantum field theory of disorder in the early Universe which goes beyond the perturbative treatment. There, we simply pointed out the correspondence of non-equilibrium quantum field theories and cosmological *in-in* path integral. We finally arrived to the conclusion that if we are to consider studying disorder in microscopic early Universe, non-equilibrium quantum field theories may provide the correct theoretical framework.

Throughout this paper we tried to take steps towards studying disorder in cosmology, as it is defied in condensed matter theories. Although our work was on mainly considering formalisms, we have seen that early Universe cosmology and specifically inflation, indeed made it possible for us to introduce and calculate for the mechanism of disorder. Considering the classical methods studied in this work, it would be interesting to see whether if we can *improve* the variational approach in estimating the free energy in a given scenario. Considering quantum physics, it should be possible to *extend* the existing models of inflation by simply introducing disorder to models' parameters and calculate the corrections from this effect. Finally for a complete study of disorder in the early Universe, we must work towards extending the non-equilibrium field theories to realise the dynamics of inflation in de Sitter space. We have seen in the replica applications of these theories that inflationary scenario can be studied with similar condensed matter tools. It would be interesting to see to what extend this correspondence between disorder in condensed matter formalisms and early Universe physics can be established.

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